

Topological Cohen–Macaulay criteria for monomial ideals

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Introduction

Scattered over the past few years have been several occurrences of simplicial complexes whose topological behavior characterize the Cohen–Macaulay property for quotients of polynomial rings by arbitrary (not necessarily squarefree) monomial ideals. It is unclear whether researchers thinking about this topic have, to this point, been aware of the full spectrum of related developments. Therefore, the purpose of this survey is to gather the developments into one location, with self-contained proofs, including direct combinatorial topological connections between them.

Four families of simplicial complexes are defined in reverse chronological order: via distraction, the Čech complex, Alexander duality, and then the Koszul complex. Each comes with historical remarks and context, including forays into Stanley decompositions, standard pairs, A -hypergeometric systems, cellular resolutions, the Čech hull, polarization, local duality, and duality for \mathbb{Z}^n -graded resolutions.

Results or definitions appearing here for the first time include the general categorical definition of cellular complex in Definition 3.2, as well as the statements and proofs of Lemmas 3.9 and 3.10, though these are very easy. The characterization of exponent simplicial complexes in Corollary 2.9 might be considered new; certainly the consequent connection in Theorem 4.1 to the Čech simplicial complexes is new, as is the duality in Theorem 6.3 between these and the dual Čech simplicial complexes. The geometric connection between distraction and local cohomology in Theorem 4.7, and its consequences in Section 5, generalize and refine results from [BeMa08], in addition to providing commutative proofs. Finally, the connection between Čech and Koszul simplicial complexes in Lemma 7.2 and Corollary 7.8, as well as its general \mathbb{Z}^n -graded version in Theorem 7.7, appear to be new.

1. Monomial ideals and simplicial complexes

Throughout this article, I is an ideal in the polynomial ring $S = \mathbb{k}[x_1, \dots, x_n]$. The coefficient field \mathbb{k} is assumed arbitrary unless explicitly stated otherwise. Each

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monomial in S can be expressed uniquely as $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ for some vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ of n nonnegative integers. More generally, Laurent monomials in $S[x_1^{-1}, \dots, x_n^{-1}]$ can be expressed as $\mathbf{x}^{\mathbf{a}}$ for vectors $\mathbf{a} \in \mathbb{Z}^n$ of arbitrary integers.

For any subset $F \subseteq \{1, \dots, n\}$ of indices, let $\mathbb{N}^F \subseteq \mathbb{Z}^F$ denote the *coordinate subspace* of vectors whose entries are zero outside of those indexed by F . This means that \mathbb{Z}^F consists of the vectors whose *support*, by definition the set $\text{supp}(\mathbf{a})$ of indices i for which a_i is nonzero, is contained in F . Identify the set F of indices with its characteristic vector in \mathbb{N}^n , which is the vector with support F and all nonzero entries equal to 1. Thus every squarefree monomial is $\mathbf{x}^F = \prod_{i \in F} x_i$ for some index set F . The inverse of such a monomial is \mathbf{x}^{-F} , so $S[\mathbf{x}^{-F}] = S[x_i^{-1} \mid i \in F]$ is a *monomial localization* of S , which we denote by S_F . In addition, let

$$\mathfrak{m}^F = \langle \mathbf{x}_F \rangle = \langle x_i \mid i \in F \rangle$$

be the ideal generated by the variables indexed by F . Thus $S/\mathfrak{m}^F = \mathbb{k}[\mathbf{x}_{\bar{F}}]$ is a polynomial ring in variables $\mathbf{x}_{\bar{F}}$ indexed by the complementary set $\bar{F} = \{1, \dots, n\} \setminus F$.

For any ideal $I \subseteq S$, define the *zero set*

$$Z(I) = \{\alpha \in \mathbb{k}^n \mid f(\alpha) = 0 \text{ for all } f \in I\}.$$

If I is generated by monomials, for example, then $Z(I)$ is a union of coordinate subspaces of \mathbb{k}^n . Conversely, if any union Z of coordinate subspaces is given, then the ideal $I(Z)$ of polynomials vanishing on Z is a monomial ideal. (When \mathbb{k} is finite, this statement must be interpreted in the appropriate scheme-theoretic way; but there is no harm in simply assuming that \mathbb{k} is algebraically closed—and hence infinite—for the duration of this article.) In fact, $I(Z)$ is a *squarefree* monomial ideal: its minimal monomial generators are all squarefree. On the other hand, a union Z of coordinate subspaces also gives rise to a *simplicial complex*

$$\Delta(Z) = \{F \subseteq \{1, \dots, n\} \mid \mathbb{k}^F \subseteq Z\},$$

meaning that $\Delta(Z)$ is a collection of subsets, called *faces*, that is closed under inclusion: $F \in \Delta(Z)$ and $G \subseteq F$ implies $G \in \Delta(Z)$. The assignments $Z \rightsquigarrow I(Z)$ and $Z \rightsquigarrow \Delta(Z)$ induce a bijection, called the *Stanley–Reisner correspondence*, between squarefree monomial ideals and simplicial complexes. Given a simplicial complex Δ , the corresponding monomial ideal I_Δ is its *Stanley–Reisner ideal* (also known as its *face ideal*); given a squarefree monomial ideal, Δ_I is its *Stanley–Reisner complex*.

Background on squarefree monomial ideals and simplicial complexes can be found in Chapter II of Stanley’s book [Sta96], or in Chapter 5 of the book by Bruns and Herzog [BrHe93], or in Chapter 1 of the book by Miller and Sturmfels [MiSt05]. The starting point for this survey is the following.

DEFINITION 1.1. A simplicial complex Δ is *Cohen–Macaulay over \mathbb{k}* if the Stanley–Reisner ring $\mathbb{k}[\Delta]$ is Cohen–Macaulay.

The Cohen–Macaulay condition on Δ is combinatorial, as discovered by Reisner.

THEOREM 1.2 (Reisner’s criterion, [Rei76]). *A simplicial complex Δ is Cohen–Macaulay over \mathbb{k} if and only if, for every face $F \in \Delta$, the subcomplex*

$$\text{link}(F, \Delta) = \{G \in \Delta \mid G \cup F \in \Delta \text{ and } G \cap F = \emptyset\},$$

known as the link of F inside Δ , has reduced cohomology

$$\tilde{H}^i(\text{link}(F, \Delta); \mathbb{k}) = 0 \quad \text{for } i \neq \dim(\Delta) - |F|.$$

PROOF. This follows from Theorem 3.1, below; alternatively, see [Sta96, Corollary II.4.2], [BrHe93, Corollary 5.3.9], or [MiSt05, Section 13.5.2]. \square

REMARK 1.3. As combinatorial as Reisner’s criterion may appear, in reality it is topological: it is equivalent to the vanishing of all but the top reduced cohomology of Δ over \mathbb{k} and the top *local cohomology* over \mathbb{k} near every point p in the geometric realization $|\Delta|$; that is, the relative cohomology with coefficients in \mathbb{k} of the pair $(|\Delta|, |\Delta| \setminus \{p\})$ [Sta96, Proposition II.4.3].

REMARK 1.4. Note how the field \mathbb{k} enters into the definition: although the presentation of the ring $\mathbb{k}[\Delta]$ looks the same for every field \mathbb{k} , the Cohen–Macaulay condition can depend on the characteristic of \mathbb{k} (but nothing else—all fields with the same characteristic behave just like the prime field of that characteristic). This is reflected in Reisner’s criterion, where the simplicial complex is independent of \mathbb{k} , but its cohomology is taken with coefficients in \mathbb{k} .

In recent years, Reisner’s topological criterion for the Cohen–Macaulay condition has been generalized to settings where the monomial ideal I is no longer required to be squarefree. The essential framework is to define not one simplicial complex, but a family of them indexed by subsets of \mathbb{Z}^n , or \mathbb{k}^n in some cases, and then connect the Cohen–Macaulayness of members in this family to the homological algebra of I , or to the geometry of closely related ideals. The purpose of this survey is to demonstrate direct equivalences between various families of simplicial complexes arising this way.

2. Distracting arrangements and exponent complexes

The first step is to produce a single family of simplicial complexes whose Cohen–Macaulayness reflects that of a given monomial ideal I , which we fix from now on.

DEFINITION 2.1. The monomials in S outside of I are called *standard monomials*. Write $\Lambda_I \subseteq \mathbb{N}^n$ for the set of exponent vectors on the standard monomials of I .

LEMMA 2.2. *The standard monomials form a \mathbb{k} -vector space basis for S/I .* \square

The geometry of Λ_I closely reflects the geometry of S/I . When I is squarefree, for example, Λ_I is the union of the coordinate subspaces \mathbb{N}^F for the faces $F \in \Delta_I$ in the Stanley–Reisner simplicial complex, while the zero set $Z(I)$ is the union of the coordinate subspaces \mathbb{k}^F for the faces $F \in \Delta_I$. In general, Λ_I “looks” like the zero scheme of I , rather than the zero set $Z(I)$. Nonetheless, the transition from standard monomials to subspace arrangement generalizes to arbitrary monomial ideals, as long as \mathbb{k} has characteristic zero (or characteristic bigger than the greatest exponent on any minimal generator of I). For simplicity, we shall work with the field $\mathbb{k} = \mathbb{C}$ of complex numbers.

DEFINITION 2.3. The *distracting arrangement* V_I of the monomial ideal I is the Zariski closure $Z(I(\Lambda_I)) \subseteq \mathbb{C}^n$ of the subset $\Lambda_I \subseteq \mathbb{N}^n \subseteq \mathbb{C}^n$.

Taking this Zariski closure has the same feel as the squarefree case. For example, if a lattice point lies along a line containing infinitely many points in Λ_I , then the whole line is contained in V_I . More generally, V_I has the following description.

LEMMA 2.4. *Λ_I can be expressed as a finite disjoint union of translates $\mathbf{a} + \mathbb{N}^F$ of coordinate subspaces of \mathbb{N}^n . For any such decomposition, the distracting arrangement is the corresponding union of translates $\mathbf{a} + \mathbb{C}^F$ of coordinate subspaces of \mathbb{C}^n .*

The proof, and a fair bit of the rest of this survey, uses \mathbb{Z}^n -gradings; see [MiSt05, Chapter 8]. In particular, $S = \bigoplus_{\mathbf{b} \in \mathbb{Z}^n} S_{\mathbf{b}}$, where $S_{\mathbf{b}}$ is the 1-dimensional vector space spanned by the monomial $\mathbf{x}^{\mathbf{b}}$ if $\mathbf{b} \in \mathbb{N}^n$, and $S_{\mathbf{b}} = 0$ if $\mathbf{b} \in \mathbb{Z}^n \setminus \mathbb{N}^n$.

PROOF. Every \mathbb{Z}^n -graded S -module has a monomial associated prime [MiSt05, Proposition 8.11], and hence a \mathbb{Z}^n -graded submodule that is a \mathbb{Z}^n -graded translate $\mathbf{x}^{\mathbf{a}} \cdot S/\mathfrak{m}^F$ of a quotient of S modulo a monomial prime. By Noetherian induction, S/I possesses a filtration by \mathbb{Z}^n -graded submodules whose associated graded pieces have this form. The first sentence is thus a consequence of Lemma 2.2, which implies that the monomials with exponent vectors in $\mathbf{a} + \mathbb{N}^F$ form a basis for $\mathbf{x}^{\mathbf{a}} \cdot S/\mathfrak{m}^F$. The second sentence follows from the first, since Zariski closure commutes with finite unions. \square

Unions as in Lemma 2.4 are called *Stanley decompositions* of S/I . They are relatively easy to come by, although finding ones with good properties is difficult: Stanley's conjecture [Sta82], which posits that one exists in which the minimum size of any appearing F is at least the depth of S/I , is still open; see [HVZ08] for recent work on this topic and more background.

Locally near every point in \mathbb{C}^n , the distracting arrangement V_I looks like a coordinate subspace arrangement. Here is a more precise statement; see later in this section for history and context.

DEFINITION 2.5. For $\alpha \in \mathbb{C}^n$, let $Z_{\alpha} \subseteq V_I$ be the union of the components of V_I passing through α . Then $-\alpha + Z_{\alpha}$ is a coordinate subspace arrangement in \mathbb{C}^n , and $I(-\alpha + Z_{\alpha})$ is the Stanley–Reisner ideal of the *exponent simplicial complex* $E\Delta_{\alpha}$.

REMARK 2.6. Given a fixed set of monomial generators, the ideal $I \subseteq S$ can have different homological properties depending on the characteristic of \mathbb{k} ; see Remark 1.4. The simplicial complexes $E\Delta_{\alpha}$ for $\alpha \in \mathbb{C}^n$, on the other hand, by definition do not make reference to an arbitrary field \mathbb{k} . It may therefore come as a surprise that the topological invariants of $E\Delta_{\alpha}$ for $\alpha \in \mathbb{Z}^n$ control the homological properties of S/I for any field \mathbb{k} ; see Theorems 3.8, 4.1, and 4.7, as well as Corollary 5.1. But this is analogous to the squarefree case, where the Stanley–Reisner complex is defined without reference to \mathbb{k} , whereas its homological invariants depend on \mathbb{k} ; it justifies the restriction to $\mathbb{k} = \mathbb{C}$ in Definition 2.3.

Although there is no known canonical way to choose a Stanley decomposition, a measure of uniqueness can be imposed by dispensing with the disjointness hypothesis. As a matter of terminology, a *cover* of any set Λ is a family of subsets whose union is Λ . One cover *dominates* another if every subset from the second cover is contained in a subset from the first.

PROPOSITION 2.7. Λ_I has a unique cover by translates $\mathbf{a} + \mathbb{N}^F$ of coordinate subspaces of \mathbb{N}^n dominating every other such cover. Each standard pair $\mathbf{a} + \mathbb{N}^F$ in this standard pair decomposition of Λ_I is the intersection with Λ_I of an irreducible component $\mathbf{a} + \mathbb{C}^F$ of the distracting arrangement V_I , and \mathbf{a} has support disjoint from F .

PROOF. Every component $\mathbf{a} + \mathbb{C}^F$ of V_I is the closure of a translated orthant $\mathbf{a} + \mathbb{N}^F$ for some $\mathbf{a} \in \mathbb{N}^n$, by Lemma 2.4. Observe that $(\mathbf{a} + \mathbb{C}^F) \cap \mathbb{N}^n = \mathbf{a}_F + \mathbb{N}^F$, where \mathbf{a}_F is obtained from \mathbf{a} by setting to 0 the coordinates of \mathbf{a} indexed by \bar{F} . Once Λ_I contains $\mathbf{a} + \mathbb{N}^F$, it contains $\mathbf{a}_F + \mathbb{N}^F$, because the standard monomials are closed under divisibility: every divisor of every standard monomial is standard. \square

Standard pair decompositions were introduced by Sturmfels, Trung, and Vogel [STV95, Section 3], for the reason that the number of times F appears equals the multiplicity of the prime ideal $\mathfrak{m}^{\bar{F}}$ in the primary decomposition of I . Later, standard pairs became the device through which the exponent simplicial complexes $E\Delta_\alpha$ were connected to the Cohen–Macaulay property.

REMARK 2.8. The earliest context in which exponent simplicial complexes seem to have appeared explicitly, in the form of Definition 2.5, was in the noncommutative algebra of multivariate hypergeometric systems [SST00]. The question of characterizing the Cohen–Macaulay condition had arisen naturally. The combinatorial commutative context there was more general than the one here (it was for affine semigroup rings), but it led back to distractions, nonetheless, as follows.

Associated to an affine semigroup is a family, parametrized by \mathbb{C}^d , of systems of linear partial differential equations with polynomial coefficients [GGZ87, GKZ89]. Based on observations by Gelfand, Kapranov, and Zelevinsky, along with subsequent work by Adolphson [Ado94], Sturmfels conjectured that the systems in this family all have the same number of linearly independent solutions precisely when the corresponding affine semigroup ring over the complex numbers is Cohen–Macaulay.

This conjecture was proved by Matusevich, Miller, and Walther [MMW05] by introducing functorial noncommutative Koszul-like homological methods for \mathbb{Z}^d -graded modules over semigroup rings: the rank of the solution space jumps whenever the parameter in \mathbb{C}^d lies in a certain subspace arrangement. The arrangement is a quasidegree set as in Corollary 5.10, coming from local cohomology just as distractions come from monomial quotients. In particular, the arrangement is nonempty—so there exists at least one rank jumping parameter in \mathbb{C}^d —precisely when there is nonzero local cohomology to indicate the failure of Cohen–Macaulayness.

Berkesch and Matusevich [BeMa08] used an extension [DMM08] of these homological methods to more general multigraded modules over polynomial rings to see what the Cohen–Macaulay characterization by rank jumps says about quotients by monomial ideals. Their conclusions were based on the fact that ranks of systems of differential equations arising from monomial ideals are controlled by the geometry of their distractions, from which Berkesch and Matusevich defined the exponent simplicial complexes using standard pairs. Sections 4 and 5 contain details about the results in [BeMa08]; for now, the reader is encouraged to consult that article for colorful illustrations of distracting arrangements and exponent simplicial complexes.

Proposition 2.7 results in an alternate characterization of exponent complexes that will be useful in the comparisons with other simplicial complexes to come.

COROLLARY 2.9. *A subset $F \subseteq \{1, \dots, n\}$ is a maximal face (that is, a facet) of the exponent complex $E\Delta_\alpha$ of I if and only if the set $(\alpha + \mathbb{C}^F) \cap \Lambda_I$ of lattice points is dense in $\alpha + \mathbb{C}^F$ and F is maximal with this property.*

PROOF. F is a facet of Δ_α if and only if $\alpha + \mathbb{C}^F$ is an irreducible component of the distracting arrangement V_I , by Definition 2.5. If $\alpha + \mathbb{C}^F$ is an irreducible component of V_I , then $(\alpha + \mathbb{C}^F) \cap \Lambda_I$ is a dense subset of it by Proposition 2.7, and F is maximal with this property because otherwise $\alpha + \mathbb{C}^F$ would be strictly contained in a component $\alpha + \mathbb{C}^{F'}$ for some other face F' . On the other hand, if $(\alpha + \mathbb{C}^F) \cap \Lambda_I$ is dense in $\alpha + \mathbb{C}^F$, then $\alpha + \mathbb{C}^F \subseteq V_I$ by Definition 2.3; maximality of F guarantees that $\alpha + \mathbb{C}^F$ is a component, given that each component of V_I is parallel to a coordinate subspace of \mathbb{C}^n , which follows from Proposition 2.7, or even Lemma 2.4. \square

3. Čech simplicial complexes

One of the many characterizations of the Cohen–Macaulay condition [MiSt05, Theorem 13.37] for a module M is by vanishing of all but the top—that is, cohomological degree $\dim(M)$ —local cohomology $H_{\mathfrak{m}}^i(M)$ supported on the maximal ideal \mathfrak{m} of S . In particular, Reisner’s criterion (Theorem 1.2) is a consequence of the following famous combinatorial formula for local cohomology.

THEOREM 3.1 (Hochster’s formula). *The \mathbb{Z}^n -graded Hilbert series of the local cohomology with maximal support of a Stanley–Reisner ring satisfies*

$$H(H_{\mathfrak{m}}^i(S/I_{\Delta}); \mathbf{x}) = \sum_{F \in \Delta} \dim_{\mathbb{k}} \tilde{H}^{i-|F|-1}(\text{link}(F, \Delta); \mathbb{k}) \prod_{j \in F} \frac{x_j^{-1}}{1 - x_j^{-1}}.$$

PROOF. This can be deduced from Theorem 3.8, below; alternatively, see [Sta96, Theorem II.4.1], [BrHe93, Theorem 5.3.8], or [MiSt05, Theorem 13.13]. \square

The proof of this statement is remarkably straightforward: the Čech complex of S/I_{Δ} is a complex of \mathbb{Z}^n -graded S -modules, whose graded piece in degree \mathbf{a} is the cochain complex of the appropriate link in Δ , with a homological shift. This argument is a quintessential example of the “cellular” technique that pervades a significant portion of combinatorial commutative algebra in recent years. Quite generally, this technique begins with a cell complex X , which could be a CW complex, a simplicial complex, a polyhedral complex, or any other desired type of cell complex. This topological cell complex X has an algebraic chain complex

$$\mathcal{C}_{\bullet} X : \quad 0 \leftarrow \bigoplus_{\text{vertices } v \in X} \mathbb{Z}_v \leftarrow \bigoplus_{\text{edges } e \in X} \mathbb{Z}_e \leftarrow \cdots \leftarrow \bigoplus_{i\text{-faces } \sigma \in X} \mathbb{Z}_{\sigma} \leftarrow \cdots$$

over the integers, where $\mathbb{Z}_{\tau} \leftarrow \mathbb{Z}_{\sigma}$ is multiplication by some integer $\text{coeff}(\sigma, \tau)$.

DEFINITION 3.2. Fix a cell complex X , an abelian category, an object J_{σ} for each face $\sigma \in X$, and a natural map $J_{\tau} \leftarrow J_{\sigma}$ for each inclusion $\tau \subseteq \sigma$ of faces. The complex

$$J_{\bullet} : \quad 0 \leftarrow \bigoplus_{\text{vertices } v \in X} J_v \leftarrow \bigoplus_{\text{edges } e \in X} J_e \leftarrow \cdots \leftarrow \bigoplus_{i\text{-faces } \sigma \in X} J_{\sigma} \leftarrow \cdots$$

is *cellular* and *supported on X* if the component $J_{\tau} \leftarrow J_{\sigma}$ is $\text{coeff}(\sigma, \tau)$ times the given natural map. A similar construction yields *cocellular* complexes, starting from the algebraic cochain complex of X and the natural maps are $J_{\tau} \rightarrow J_{\sigma}$ for $\tau \subseteq \sigma$.

REMARK 3.3. There are more variations than those mentioned in Definition 3.2. For example, we could start from the *reduced* (or *augmented*) algebraic chain or cochain complex of a simplicial complex, which includes another (co)homological degree for the empty face of X . We could also start from the algebraic relative (co)chain complex of a pair of cell complexes. Generally speaking, any complex in an abelian category constructed from underlying topological data on a cell complex, or pair of cell complexes, is referred to as a *cellular complex* of objects in that category.

EXAMPLE 3.4. The prototypical example is the algebraic cochain complex arising from Čech cohomology theory in topology. In this case, the cell complex X is a simplex, and J_{σ} is a space of functions on an intersection $\bigcap_{j \in \sigma} U_j$ of open sets in a given cover $\{U_1, \dots, U_n\}$ of a fixed topological space (usually unrelated to X).

In its \mathbb{Z}^n -graded avatar, the *Čech complex* $\check{\mathcal{C}}^{\bullet}(\mathbf{x})$ is a cocellular complex of monomial localizations of S supported on a simplex X . More precisely, X is the

simplex on $\{1, \dots, n\}$, and $J_F = S_F$ is the monomial localization $S[\mathbf{x}^{-F}]$, the natural map $S_F \rightarrow S_{F'}$ being the localization homomorphism for $F \subseteq F'$.

REMARK 3.5. Other cellular complexes have played roles in commutative algebra for well over half a century, perhaps the first being the Koszul complex [Kos50], which was originally introduced for the purpose of Lie algebra cohomology theory. The initial use of cellular complexes in commutative algebra that is identifiably combinatorial came with the Taylor complex [Tay60], which provides a free resolution of any monomial ideal in a polynomial ring. The application of Čech complexes to local cohomology of monomial ideals was carried out in the 1970's. The earliest cellular complexes supported on polyhedra that are not simplicial complexes arrived in the work of Ishida on dualizing complexes and local cohomology with maximal support over affine semigroup rings [Ish80, Ish87]; in addition, Ishida's objects J_σ were modules over rings other than polynomial rings. The notion of *cellular resolution* in combinatorial commutative algebra was codified formally for the first time in the work of Bayer and Sturmfels, also with Peeva [BaSt98, BPS98], in the context of lattice ideals and generic monomial ideals (the more inclusive definition of generic monomial ideals in current use [MSY00] was introduced later).

More general objects J_σ , including graded injectives and quotients by irreducible monomial ideals in polynomial and affine semigroup rings, began to appear in [Mil00] and [Mil02b], where topological duality for the support complexes was observed to reflect algebraic (Matlis or local) duality algebraically. All of the cellular resolutions to that point had been supported on special classes of cell complexes, notably simplicial and polyhedral complexes. With the work of Batzies and Welker [BaWe02], more flexible support complexes X began appearing, allowing the application of homotopic and Morse-theoretic cancellative approaches, which don't preserve polyhedrality. Cellular methods generally bring topological or combinatorial techniques to bear on algebraic problems, but Fløystad, with his *enriched cohomology*, has begun to realize that commutative algebra can provide additional homological insight into the cell complex X , rather than vice versa [Flø06]. Most recently, cellular techniques in combinatorial commutative algebra have given way to non-combinatorial applications where the objects J_σ do not belong to commutative algebra at all: in [JoMi08], where Definition 3.2 was mentioned in the Introduction without details, they are sheaves from scheme-theoretic algebraic geometry, or perhaps from complex analytic geometry.

These examples are idiosyncratic choices of landmarks; there have been many others between, demonstrating the strengths and limitations of cellular resolutions.

Generalizing Theorem 3.1 to the local cohomology of S/I for an arbitrary monomial ideal I involves simplicial complexes that can be characterized as follows.

DEFINITION 3.6. Given $\mathbf{a} \in \mathbb{Z}^n$, write $\mathbf{a} = \mathbf{a}_+ - \mathbf{a}_-$ as a difference of vectors in \mathbb{N}^n . The Čech simplicial complex of I in degree \mathbf{a} is the set

$$\check{C}\Delta_{\mathbf{a}} = \{F - \text{supp}(\mathbf{a}_-) \mid (S_F/IS_F)_{\mathbf{a}} = \mathbb{k}\}$$

in bijection with those F for which the localization $S/I \otimes_S S_F$ is nonzero in degree \mathbf{a} .

LEMMA 3.7. $\check{C}\Delta_{\mathbf{a}}$ is a simplicial complex.

PROOF. For $F \subseteq F'$, the homomorphism from S_F/IS_F to its image in $S_{F'}/IS_{F'}$ is the quotient homomorphism of S_F/IS_F modulo the kernel of the localization

that inverts $\mathbf{x}^{F'-F}$. Therefore, once S_F is nonzero in degree \mathbf{a} , the homomorphism $(S_F/IS_F)_{\mathbf{a}} \rightarrow (S_{F'}/IS_{F'})_{\mathbf{a}}$ is a quotient map. The final observation to make is that S_F is nonzero in degree \mathbf{a} precisely when F contains $\text{supp}(\mathbf{a}_-)$. \square

THEOREM 3.8. *The Čech simplicial complexes $\check{C}\Delta_{\mathbf{a}}$ of I compute local cohomology:*

$$H_{\mathfrak{m}}^i(S/I)_{\mathbf{a}} = \tilde{H}^{i-|\text{supp}(\mathbf{a}_-)|-1}(\check{C}\Delta_{\mathbf{a}}; \mathbb{k}).$$

PROOF. The local cohomology $H_{\mathfrak{m}}^i(S/I)_{\mathbf{a}}$ is the \mathbb{Z}^n -degree \mathbf{a} piece of the Čech complex $S/I \otimes_S \check{C}^{\bullet}(\mathbf{x})$. Viewing the Čech complex as being cocellular as in Example 3.4, and using that $S/I \otimes_S S_F = S_F/IS_F$, the degree \mathbf{a} piece of $S/I \otimes_S \check{C}^{\bullet}(\mathbf{x})$ is the reduced cochain complex of $\check{C}\Delta_{\mathbf{a}}$ by Definition 3.6, with the faces $F - \text{supp}(\mathbf{a}_-)$ of dimension $i - |\text{supp}(\mathbf{a}_-)| - 1$ contributing to cohomological degree $i = |F|$. \square

Theorem 3.8 was recorded by Takayama [Tak05, Theorem 1], as the first step in studying locally Cohen–Macaulay monomial schemes—that is, whose local cohomology modules $H_{\mathfrak{m}}^i(S/I)$ have finite length for $i < \dim(S/I)$. Takayama also proved, using the simplicial formula, that $H_{\mathfrak{m}}^i(S/I)$ is nonzero only in degrees $\mathbf{a} \in \mathbb{Z}^n$ satisfying $\mathbf{a} \preceq \mathbf{a}_I - \mathbb{1}$, where \mathbf{a}_I is the exponent on the least common multiple of the minimal generators, $\mathbb{1} = (1, \dots, 1)$, and the relation \preceq is coordinatewise dominance. This observation admits a simple proof in the general context of arbitrary finitely generated \mathbb{Z}^n -graded modules by using \mathbb{Z}^n -graded injective resolutions (see [MiSt05, Chapter 11] for an introduction). The general statement is as follows.

LEMMA 3.9. *If the Betti numbers $\beta_{i,\mathbf{b}}(M)$ of a finitely generated \mathbb{Z}^n -graded module M lie in degrees $\mathbf{b} \preceq \mathbf{a}$, then $H_{\mathfrak{m}}^i(M)$ can only be nonzero in \mathbb{Z}^n -degrees $\preceq \mathbf{a} - \mathbb{1}$.*

PROOF. The proof consists of two observations:

- (i) the Betti number $\beta_{i,\mathbf{b}}(M)$ equals the Bass number $\mu_{n-i,\mathbf{b}-\mathbb{1}}(M)$ at the maximal ideal, as can be seen by using a Koszul complex to compute both; and
- (ii) the \mathbb{Z}^n -graded injective hull of \mathbb{k} is nonzero only in degrees from $-\mathbb{N}^n$. \square

In fact, (i) implies more, when M is graded by \mathbb{N}^n , and not just \mathbb{Z}^n .

LEMMA 3.10. *If M is \mathbb{N}^n -graded and $H_{\mathfrak{m}}^i(M)$ is finitely generated (i.e., has finite length), then the \mathbb{Z}^n -graded translate $H_{\mathfrak{m}}^i(M)(-\mathbb{1})$ up by $(1, \dots, 1)$ is also \mathbb{N}^n -graded.*

PROOF. The Čech hull [Mil00, Definition 2.7] (or see [MiSt05, Definition 13.32]) of any \mathbb{Z}^n -graded module N is the \mathbb{Z}^n -graded module $\check{C}N$ whose degree \mathbf{b} piece is $(\check{C}N)_{\mathbf{b}} = N_{\mathbf{b}_+}$ for $\mathbf{b} \in \mathbb{Z}^n$. The Čech hull is an exact functor on \mathbb{Z}^n -graded modules because this is measured degree-by-degree. It is easy to show that the Čech hull of any indecomposable \mathbb{Z}^n -graded injective module E is either zero or E , the latter precisely when the \mathbb{Z}^n -graded degree piece $E_{\mathbf{0}}$ is nonzero [Mil00, Lemma 4.25]. It follows immediately from (i) that the minimal injective resolution of $M(-\mathbb{1})$ —and hence all of the local cohomology of $M(-\mathbb{1})$ —is fixed by the Čech hull. The previous sentence is what the statement of the lemma should really say, since any finitely generated module fixed by the Čech hull must vanish in all degrees $\notin \mathbb{1} + \mathbb{N}^n$. \square

REMARK 3.11. Rahimi generalized Theorem 3.8 to a combinatorial formula for the local cohomology of S/I supported at an arbitrary prime monomial ideal [Rah07]. It remains open to find a combinatorial formula for the local cohomology of S/I supported at an arbitrary (may as well be squarefree) monomial ideal.

4. Local cohomology vs. distraction

Thus far we have seen two families of simplicial complexes: the exponent (Definition 2.5) and Čech (Definition 3.6) complexes. They were introduced in radically different contexts, the former being the geometry of distractions, and the latter being the graded algebra of local cohomology; but they are very closely related.

THEOREM 4.1. *The exponent simplicial complex $E\Delta_{\mathbf{a}}$ of I in degree $\mathbf{a} \in \mathbb{Z}^n$ is a cone with apex $\text{supp}(\mathbf{a}_-)$, and the Čech simplicial complex $\check{C}\Delta_{\mathbf{a}}$ is the link of the apex:*

$$\check{C}\Delta_{\mathbf{a}} = \text{link}(\text{supp}(\mathbf{a}_-), E\Delta_{\mathbf{a}}).$$

More generally, for $\alpha \in \mathbb{C}^n$, define $\alpha' \in \mathbb{Z}^n$ by setting to -1 all coordinates of α that lie outside of \mathbb{N} . Then $E\Delta_{\alpha}$ is a cone with apex α'_- and $\check{C}\Delta_{\alpha'} = \text{link}(\alpha'_-, E\Delta_{\alpha})$.

PROOF. First observe that $\check{C}\Delta_{\mathbf{a}'} = \check{C}\Delta_{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{Z}^n$, since the variables x_i for $i \in \text{supp}(\mathbf{a}_-) = \mathbf{a}'_-$ are units on S_F/IS_F whenever $F \supseteq \text{supp}(\mathbf{a}_-)$. Hence the second sentence is indeed more general than the first.

LEMMA 4.2. *$E\Delta_{\alpha'} = E\Delta_{\alpha}$ for all $\alpha \in \mathbb{C}^n$. More generally, if β is obtained from $\alpha \in \mathbb{C}^n$ by changing some nonnegative integer entries of α to lie outside of \mathbb{N} , then $E\Delta_{\beta} = \text{star}(\text{supp}(\alpha - \beta), E\Delta_{\alpha})$, where for any simplicial complex Δ and face F ,*

$$\text{star}(F, \Delta) = \{F' \in \Delta \mid F \cup F' \in \Delta\}.$$

PROOF. The first sentence follows from the second by expressing both as stars in $E\Delta_{\alpha_0}$, where α_0 is obtained from α by setting to 0 all entries of α lying outside of \mathbb{N} .

For F to be a facet of $E\Delta_{\alpha}$, it must by Corollary 2.9 contain the set of indices i for which α_i lies outside of \mathbb{N} , or else the affine subspace $\alpha + \mathbb{C}^F$ fails to intersect \mathbb{N}^n , let alone being the closure of a set of lattice points therein. Therefore the facets of $E\Delta_{\beta}$ are precisely the facets of $E\Delta_{\alpha}$ that contain $\text{supp}(\alpha - \beta)$. \square

The lemma reduces Theorem 4.1 to the claim that $\check{C}\Delta_{\mathbf{a}} = \text{link}(\mathbf{a}_-, E\Delta_{\mathbf{a}})$ when $\mathbf{a}' = \mathbf{a}$. For this, it is enough by Corollary 2.9 to show, when $\mathbf{a}_- \subseteq F$, that $(S_F/IS_F)_{\mathbf{a}} = \mathbb{k}$ if and only if $(\mathbf{a} + \mathbb{C}^F) \cap \Lambda_I$ is dense in $\mathbf{a} + \mathbb{C}^F$; but both of these, given $\mathbf{a}_- \subseteq F$, are equivalent to $\mathbf{x}^{\mathbf{a}+}\mathbb{C}[\mathbf{x}_F]$ being a subquotient of S/I . \square

REMARK 4.3. Berkesch and Matusевич observed (cf. Remark 2.8) that the cohomology of the exponent simplicial complexes of I controls the Cohen–Macaulay property. Their precise statement along these lines [BeMa08, Theorem 3.12] is the final claim in Corollary 5.1, below. However, they did not explicitly mention that the Cohen–Macaulay property for all exponent simplicial complexes is equivalent to the Cohen–Macaulay property for the distracting arrangement V_I , because their definition of $E\Delta_{\alpha}$ was combinatorial, in contrast to the equivalent geometric definition in Section 2 here. Nevertheless, the next main result here, Theorem 4.7, which provides a precise geometric criterion for the failure of the Cohen–Macaulay condition at a point of V_I , is inspired by a similar—but weaker—result of Berkesch and Matusевич; see Corollary 5.10.

DEFINITION 4.4. Given a \mathbb{Z}^n -graded module N , its *true degree set* is the set

$$\text{tdeg}(N) = \{\mathbf{a} \in \mathbb{Z}^n \mid N_{\mathbf{a}} \neq 0\}$$

of \mathbb{Z}^n -graded degrees where N is nonzero. The *quasidegree set* $\text{qdeg}(N)$ is the Zariski closure of $\text{tdeg}(N)$ inside of \mathbb{C}^n . Let $\text{qdeg}_{\mathbb{R}}(N) = \text{qdeg}(N) \cap \mathbb{R}^n$ be the *real quasidegree set* of N .

EXAMPLE 4.5. The true degree set of S/I is $\text{tdeg}(S/I) = \Lambda_I$ from Definition 2.1. The quasidegree set of S/I is the distracting arrangement V_I from Definition 2.3.

REMARK 4.6. Quasidegree sets and distracting arrangements in \mathbb{C}^n are varieties over \mathbb{C} , so it makes sense to say they are Cohen–Macaulay. Their intersections with \mathbb{R}^n are varieties over \mathbb{R} , but are also topological spaces. Using Remark 1.3, it therefore makes sense to say that quasidegree sets or distracting arrangements over \mathbb{R} are Cohen–Macaulay over a field that might not be \mathbb{C} . That said, the following result is true—with the same proof—if \mathbb{R} and \mathbb{k} are replaced by \mathbb{C} throughout.

THEOREM 4.7. *If $\dim(S/I) = d$, the real distracting arrangement $V_I(\mathbb{R}) = V_I \cap \mathbb{R}^n$ is not Cohen–Macaulay over \mathbb{k} near $\alpha \in \mathbb{R}^n \Leftrightarrow \alpha \in \text{qdeg}_{\mathbb{R}}(H_{\mathfrak{m}}^i(S/I))$ for some $i < d$.*

PROOF. The arrangement $V_I(\mathbb{R})$ fails to be Cohen–Macaulay over \mathbb{k} near $\alpha \in \mathbb{R}^n$ if and only if $E\Delta_{\alpha}$ fails to be Cohen–Macaulay over \mathbb{k} . Noting that $\text{link}(F, E\Delta_{\alpha})$ remains a cone unless F contains α'_- (Theorem 4.1), this occurs precisely when there exist a face $F \in E\Delta_{\alpha}$ with $F \supseteq \alpha'_-$ and a cohomological degree $i < d$ such that

$$\tilde{H}^{i-|F|-1}(\text{link}(F, E\Delta_{\alpha}); \mathbb{k}) \neq 0$$

by Reisner’s criterion (Theorem 1.2). On the other hand, if $\mathbf{a} = \alpha_F$ is the result of setting to -1 all coordinates of α indexed by F , then $\text{link}(F, E\Delta_{\alpha}) = \text{link}(F, E\Delta_{\mathbf{a}})$ by Lemma 4.2. But $\text{link}(F, E\Delta_{\mathbf{a}}) = \check{C}\Delta_{\mathbf{a}}$ by Theorem 4.1. Furthermore, $\check{C}\Delta_{\mathbf{a}} = \check{C}\Delta_{\mathbf{b}}$ for all $\mathbf{b} \in \mathbf{a} - \mathbb{N}^F$ by the first sentence of the proof of Theorem 4.1. Stringing these statements together, and applying Theorem 3.8, we find that $V_I(\mathbb{R})$ fails to be Cohen–Macaulay over \mathbb{k} at α if and only if

(*) there exist $F \supseteq \alpha'_-$ and $i < d$ with $\mathbf{b} \in \text{tdeg}(H_{\mathfrak{m}}^i(S/I))$ for all $\mathbf{b} \in \alpha_F - \mathbb{N}^F$.

This condition certainly implies that $\alpha \in \text{qdeg}_{\mathbb{R}} H_{\mathfrak{m}}^i(S/I)$.

For the converse, assume $i < d$ with $\alpha \in \text{qdeg}_{\mathbb{R}}(H_{\mathfrak{m}}^i(S/I))$. Then $\alpha + \mathbb{R}^F$ contains a dense subset of points in $\text{tdeg}(H_{\mathfrak{m}}^i(S/I))$ for some F and $i < d$. This set F is forced to contain the set of indices where α is not an integer, or else $\alpha + \mathbb{R}^F$ wouldn’t contain any lattice points at all. Replacing α by some point in $\alpha + \mathbb{R}^F$ with $\alpha_i = -1$ for $i \in F$ (this changes neither α_F nor α'_-), we may as well assume $\alpha \in \mathbb{Z}^n$. Lemma 3.9 implies that $\text{tdeg}(H_{\mathfrak{m}}^i(S/I)) \cap (\alpha - \mathbb{N}^F)$ contains a lattice point \mathbf{b} , since the Zariski closure of the complementary set $\text{tdeg}(H_{\mathfrak{m}}^i(S/I)) \cap (\alpha - (\mathbb{Z}^F \setminus \mathbb{N}^F))$ of lattice points in $\alpha + \mathbb{Z}^F$ has dimension less than $|F|$. Theorems 3.8 and 4.1 imply that $\tilde{H}^{i-|\mathbf{b}'_-|-1}(\text{link}(\mathbf{b}'_-, E\Delta_{\mathbf{b}}); \mathbb{k}) \neq 0$. Given this nonvanishing, we might as well replace F with \mathbf{b}'_- , which contains α'_- by construction, proving (*). \square

REMARK 4.8. The analysis in Theorem 4.7 and its proof comes down to the fact that if some exponent simplicial complex $E\Delta_{\alpha}$ fails to be Cohen–Macaulay, then some link in $E\Delta_{\alpha}$ has nonzero reduced cohomology in cohomological degree less than its dimension. But the Čech simplicial complex $\check{C}\Delta_{\alpha}$, which is indeed a link in $E\Delta_{\alpha}$, might not itself have nonzero non-top reduced cohomology. Thus, even if you’re standing at a lattice point α whose simplicial complex fails Cohen–Macaulayness, you might not get nonvanishing local cohomology in that particular degree; instead, you might have to travel along a component of $\text{qdeg}(H_{\mathfrak{m}}^i(S/I))$, thereby taking a link, to get to a lattice point whose Čech simplicial complex does indeed have nonzero non-top local cohomology. That’s the point, really: the difference between the distracting arrangement V_I and $\text{qdeg}(H_{\mathfrak{m}}^i(S/I))$ is the difference between *possessing* a face whose link witnesses failure of Cohen–Macaulayness and actually *choosing* one.

5. Geometric Cohen–Macaulay criteria

The equivalence between exponent and Čech simplicial complexes in Theorem 4.1, and the geometric relation between the distracting arrangement and the degrees of nonvanishing local cohomology in Theorem 4.7, give rise to a number of Cohen–Macaulay criteria. Here is the first, already discussed in Remark 4.3.

COROLLARY 5.1. *S/I is Cohen–Macaulay \Leftrightarrow the real distracting arrangement $V_I(\mathbb{R})$ is Cohen–Macaulay over \mathbb{k} . In fact, S/I is Cohen–Macaulay of dimension $d \Leftrightarrow$ the exponent complexes $E\Delta_\alpha$ for $\alpha \in \mathbb{C}^n$ are Cohen–Macaulay over \mathbb{k} of dimension $d-1$.*

PROOF. By the well-known local cohomology criterion for Cohen–Macaulayness (see [MiSt05, Theorem 13.37.9], for example), Theorem 4.7, and Definition 2.5,

$$\begin{aligned} S/I \text{ is Cohen–Macaulay} &\Leftrightarrow H_{\mathfrak{m}}^i(S/I) = 0 \text{ for all } i < d \\ &\Leftrightarrow \text{qdeg}_{\mathbb{R}}(H_{\mathfrak{m}}^i(S/I)) \text{ is empty for all } i < d \\ &\Leftrightarrow V_I(\mathbb{R}) \text{ is Cohen–Macaulay over } \mathbb{k} \text{ at every point } \alpha \in \mathbb{R}^n \\ &\Leftrightarrow E\Delta_\alpha \text{ is Cohen–Macaulay over } \mathbb{k} \text{ at every point } \alpha \in \mathbb{R}^n. \end{aligned}$$

The result now follows from Lemma 4.2, which implies that $\{E\Delta_\alpha \mid \alpha \in \mathbb{C}^n\} = \{E\Delta_{\mathbf{a}} \mid \mathbf{a} \in \mathbb{Z}^n\}$; that is, no new exponent simplicial complexes arise by considering $\alpha \in \mathbb{C}^n$ instead of $\alpha \in \mathbb{R}^n$. \square

This connection between the geometry of distracting arrangements and homological properties of monomial ideals can also be seen to arise from elementary algebraic manipulation of monomials.

DEFINITION 5.2. For a variable t and $a \in \mathbb{N}$, the *distraction* of the pure power t^a is

$$\widetilde{t^a} = t(t-1)(t-2)\cdots(t-a+1),$$

a polynomial of degree a ; the distraction of $1 = t^0$ is defined to equal 1. For $\mathbf{a} \in \mathbb{N}^n$, the *distraction* of the monomial $\mathbf{x}^{\mathbf{a}} \in S$ is the product

$$\widetilde{\mathbf{x}^{\mathbf{a}}} = \prod_{i=1}^n \widetilde{x_i^{a_i}}$$

of the distractions of its pure-power factors $x_i^{a_i}$. For any monomial ideal $I \subseteq S$, the *distraction* of I is the ideal \widetilde{I} generated by the distractions of the monomials in I .

EXAMPLE 5.3. The distraction of $I = \langle x^3yz^2, y^2z^4 \rangle$ is

$$\widetilde{I} = \langle x(x-1)(x-2)yz(z-1), y(y-1)z(z-1)(z-2)(z-3) \rangle.$$

LEMMA 5.4. \widetilde{I} is generated by the distractions of the minimal generators of I . \square

The definitions allow for an algebraic version of Corollary 5.1.

COROLLARY 5.5. *If $\mathbb{k} = \mathbb{C}$, then S/I is Cohen–Macaulay \Leftrightarrow the distraction S/\widetilde{I} is.*

PROOF. S/\widetilde{I} is the complex coordinate ring of the distracting arrangement V_I . \square

Why should the algebraic formulation of distraction lead one to think that it detects Cohen–Macaulayness? It ought to because regular sequences connect a single object, the polarization, to both S/I and S/\widetilde{I} . This connection has been observed before; it was noted explicitly in the Introduction to [BCR05], for instance.

DEFINITION 5.6. For a variable t and $a \in \mathbb{N}$, the *polarization* of the pure power t^a is $t'_a = t_0 t_1 t_2 \cdots t_{a-1}$, a squarefree monomial in a new variables $t' = t_0, \dots, t_{a-1}$. For

$\mathbf{a} \in \mathbb{N}^n$, the *polarization* of the monomial $\mathbf{x}^{\mathbf{a}} \in S$ is the product

$$\mathbf{x}'_{\mathbf{a}} = \prod_{i=1}^n (x'_i)_{a_i} \in S'$$

of the polarizations of its pure-power factors $x_i^{a_i}$, where S' is a polynomial ring in variables x_{ij} for $i = 1, \dots, n$ and $j = 0, \dots, a_i - 1$. The *polarization* $I' \subseteq S'$ of a monomial ideal $I \subseteq S$ is generated by the polarizations of its minimal generators.

EXAMPLE 5.7. The polarization of $I = \langle x^3yz^2, y^2z^4 \rangle$ is

$$I' = \langle x_0x_1x_2y_0z_0z_1, y_0y_1z_0z_1z_2z_3 \rangle.$$

It is well-known that the quotient by a monomial ideal I is Cohen–Macaulay if and only if the quotient by its polarization I' is; see [MiSt05, Exercise 3.15] or [Kum08, Proposition 1.3.4]. The idea of the proof is that the elements $x_{ij} - x_{i0}$ form a regular sequence in S'/I' , and the quotient modulo this regular sequence is S/I , if one sets $x_{i0} = x_i$ for all i . This argument in fact shows that a minimal free resolution of S'/I' over S' descends to a minimal free resolution of S/I over S , so most of the homological invariants—Betti numbers, and so on—descend, as well. A similar procedure relates the polarization to the distraction, so Corollary 5.5 is not surprising from this perspective, although it is harder than one might imagine to prove it directly via regular sequences. Indeed, S/\tilde{I} is not a local ring, so the implication “ V_I is Cohen–Macaulay $\Rightarrow S/I'$ is Cohen–Macaulay” isn’t obvious.

The geometric characterization of the Cohen–Macaulay condition via distracting arrangements in Corollary 5.1 gives rise to another geometric characterization, also observed by Berkesch and Matusevich [BeMa08, Corollary 2.15]: when V_I is sliced by all translates of a sufficiently general linear subspace of dimension $n - \dim(S/I)$, the number of points in the intersection—counted with multiplicity—is constant precisely when S/I is Cohen–Macaulay. The following generalizes their result, which had additional hypotheses (such as integrality) on the projection A .

PROPOSITION 5.8. *Let $\mathbb{k} = \mathbb{C}$. If a linear map $A : \mathbb{C}^n \rightarrow \mathbb{C}^d$ takes every coordinate subspace in the zero set $Z(I)$ isomorphically to \mathbb{C}^d , then every fiber of the projection $V_I \rightarrow \mathbb{C}^d$ has Krull dimension 0. S/I is Cohen–Macaulay of dimension d exactly when every fiber has multiplicity $\deg(I)$ equal to the standard \mathbb{Z} -graded degree of I .*

PROOF. S/I is Cohen–Macaulay if and only if V_I is, by Corollary 5.1 along with Corollary 5.5 and its proof. The projection A makes V_I into a family over the smooth base \mathbb{C}^d . Since V_I is a finite union of subspaces parallel to coordinate spaces in $Z(I)$, this family is finite over \mathbb{C}^d . Under these conditions, the Cohen–Macaulay condition on V_I is equivalent to the flatness of this family (see [BrHe93, Proposition 2.2.11] or [MiSt05, Theorem 13.37.5]), which in turn is equivalent to the constancy of the multiplicity of the fibers [Har77, Proposition III.9.2(e) and Exercise II.5.8]. The constant multiplicity must be $\deg(I)$ because this is the number of dimension d irreducible components of V_I . \square

The statement from which Berkesch and Matusevich derive most of their consequences, such as the second half of Corollary 5.1 and their version of Proposition 5.8, is a coarsening [BeMa08, Corollary 2.14] of Theorem 4.7 to a \mathbb{Z}^d -graded setting. The situation is that of Proposition 5.8, except that A is required to be represented by a $d \times n$ integer matrix, also called A , whose columns span \mathbb{Z}^d and whose kernel contains no nonzero positive vectors. Such a matrix A induces a \mathbb{Z}^d -grading on S ,

in which each monomial $\mathbf{x}^{\mathbf{a}}$ has degree $A \cdot \mathbf{a} \in \mathbb{Z}^d$. Any module N that is \mathbb{Z}^d -graded has *true degrees* $\text{tdeg}_d(N) = \{\mathbf{u} \in \mathbb{Z}^d \mid N_{\mathbf{u}} \neq 0\}$ and *quasidegrees* $\text{qdeg}_d(N) = \text{Zariski closure of } \text{tdeg}_d(N) \text{ in } \mathbb{C}^d$.

LEMMA 5.9. *Fix a \mathbb{Z}^n -graded finitely generated S -module M of dimension at most d , or an artinian \mathbb{Z}^n -graded module whose \mathbb{Z}^n -graded Matlis dual has dimension at most d . Then $\text{qdeg}_d(M) = A(\text{qdeg}(M))$ is the image in \mathbb{C}^d of the \mathbb{C}^n -quasidegrees.*

PROOF. By Matlis duality, assume M is finitely generated. The set $\text{tdeg}_n(M)$ of true degrees of M admits a decomposition as a finite union of positive integer orthants having the form $\alpha + \mathbb{N}^F$, where $\alpha \in \mathbb{Z}^n$ and $|F| \leq d$. Any such decomposition yields a decomposition of $\text{tdeg}_d(M)$ as a finite union of translated semigroups $\beta + \mathbb{N}A_F$, where A_F is the set of columns of A indexed by F . The result holds because Zariski closure commutes with the application of A to such sets. \square

Here, finally, is a precise formulation of [BeMa08, Corollary 2.14].

COROLLARY 5.10. *Fix a \mathbb{Z}^d -grading by A . In the situation of Proposition 5.8, the fiber over $\gamma \in \mathbb{C}^d$ has multiplicity $> \deg(I)$ exactly when $\gamma \in \bigcup_{i=0}^{d-1} \text{qdeg}_d(H_{\mathfrak{m}}^i(S/I))$.*

PROOF. The multiplicity of the fiber $V_I(\gamma)$ over $\gamma \in \mathbb{C}^d$ is $> \deg(I)$ if and only if V_I fails to be flat near γ as a family over \mathbb{C}^d . This occurs precisely when the stalk of the coordinate ring $\mathbb{C}[V_I] = S/\tilde{I}$ near γ fails to be Cohen–Macaulay over the local ring of $\gamma \in \mathbb{C}^d$. This condition is equivalent to the failure of the Cohen–Macaulay condition for V_I locally at some point $\alpha \in \mathbb{C}^n$ mapping to γ under A . But we have already seen, in Theorem 4.7, that the set of points in V_I where the Cohen–Macaulay condition fails is the union, over $i = 0, \dots, d-1$, of the quasidegree sets $\text{qdeg}(H_{\mathfrak{m}}^i(S/I))$ in \mathbb{C}^n . The result now follows from Lemma 5.9. \square

6. Dual Čech simplicial complexes

What would a survey about topological criteria for Cohen–Macaulayness be without some mention of Alexander duality? Well, here it is, in a third family of simplicial complexes that has appeared in the literature and whose cohomology detects the Cohen–Macaulay property for an arbitrary monomial ideal I .

DEFINITION 6.1. Write $\bar{\Lambda}_I = \mathbb{N}^n \setminus \Lambda_I$ for the set of exponent vectors on the monomials in I . Given $\mathbf{b} \in \mathbb{Z}^n$, write $\mathbf{b} = \mathbf{b}_+ - \mathbf{b}_-$ as a difference of vectors in \mathbb{N}^n . The *dual Čech simplicial complex* for $\mathbf{a} \in \mathbb{Z}^n$ is

$$\check{C}\Delta^{\mathbf{a}} = \{F \subseteq \{1, \dots, n\} \mid F = \text{supp}(\mathbf{b}_-) \text{ for some } \mathbf{b} \in \mathbf{a} + \bar{\Lambda}_I\}.$$

As suggested by the nomenclature, Definition 6.1 is connected to Čech simplicial complexes by Alexander duality, as we shall see later in this section.

DEFINITION 6.2. The *Alexander dual* Δ^* of a simplicial complex Δ is characterized by the following property: $F \notin \Delta$ if and only if $\bar{F} \in \Delta^*$.

Alexander duality for simplicial complexes entered into the earliest of Hochster’s investigations into monomial commutative algebra [Hoc77], sometimes implicitly. The late 1990s saw a resurgence of explicit manifestations of Alexander duality, spurred in large part by the Eagon–Reiner theorem [ER98]: possessing a linear free resolution is dual to being Cohen–Macaulay. In what follows, only the most pertinent aspects of Alexander duality are mentioned; for additional background—both mathematical and historical—see [MiSt05, Chapter 5], including the Notes there.

The family of simplicial complexes in Definition 6.1 was identified by Mustața [Mus00, Section 2] for the purpose of a fundamental calculation, made independently by Terai [Ter99], of the local cohomology of the polynomial ring S with support on a (squarefree) monomial ideal. The resulting Hilbert series formula greatly resembles Hochster’s formula for the local cohomology of a Stanley–Reisner ring (Theorem 3.1). The relation between the Mustața–Terai formula and Hochster’s formula is a functorial effect of Alexander duality on local cohomology, known as local duality with monomial support [Mil00, Section 6]. This duality, in turn, is a combinatorial special case of Greenlees–May duality [GM92, Mil02a], which is generally an adjointness between the derived functors of completion at an ideal I and of taking support on I .

THEOREM 6.3. *If $\mathbf{a} \in \mathbb{Z}^n$ and $\mathbb{1} = (1, \dots, 1)$, then the dual Čech simplicial complex $\check{C}\Delta^{-\mathbf{a}-\mathbb{1}}$ is Alexander dual to $\check{C}\Delta_{\mathbf{a}}$ inside of the simplex on $\{1, \dots, n\} \setminus \text{supp}(\mathbf{a}_-)$.*

PROOF. $\bar{F} \in \check{C}\Delta^{-\mathbf{a}-\mathbb{1}} \Leftrightarrow \text{supp}(\mathbf{b}_-) = \bar{F}$ for some $\mathbf{b} \in -\mathbf{a} - \mathbb{1} + \bar{\Lambda}_I$

$$\begin{aligned} &\Leftrightarrow \mathbf{b}_- = -\bar{F} \text{ for some } \mathbf{b} \in -\mathbf{a} - \mathbb{1} + \bar{\Lambda}_I \\ &\Leftrightarrow -\bar{F} \in \mathbf{b} + \mathbb{Z}^F \text{ for some } \mathbf{b} \in -\mathbf{a} - \mathbb{1} + \bar{\Lambda}_I \\ &\Leftrightarrow -\bar{F} \in -\mathbf{a} - \mathbb{1} + \bar{\Lambda}_I + \mathbb{Z}^F \\ &\Leftrightarrow F - \mathbb{1} \in -\mathbf{a} - \mathbb{1} + \bar{\Lambda}_I + \mathbb{Z}^F \\ &\Leftrightarrow \mathbf{a} + F \in \bar{\Lambda}_I + \mathbb{Z}^F \\ &\Leftrightarrow \mathbf{a} \in \bar{\Lambda}_I + \mathbb{Z}^F \\ &\Leftrightarrow (IS_F)_{\mathbf{a}} = \mathbb{k} \\ &\Leftrightarrow (S_F/IS_F)_{\mathbf{a}} = 0. \\ &\Leftrightarrow F - \text{supp}(\mathbf{a}_-) \notin \check{C}\Delta_{\mathbf{a}}. \end{aligned}$$

All of the equivalences are elementary; the only one demanding verification of auxiliary data is the final line, for which it is important to check that F actually contains $\text{supp}(\mathbf{a}_-)$. This is not so bad, though: $\bar{F} \in \check{C}\Delta^{-\mathbf{a}-\mathbb{1}}$ implies that $\bar{F} \subseteq \text{supp}((-\mathbf{a} - \mathbb{1})_-) = \text{supp}((\mathbf{a} + \mathbb{1})_+)$, and this is the complement of $\text{supp}(\mathbf{a}_-)$. \square

REMARK 6.4. If Mustața was trying to compute local cohomology with support on a squarefree monomial ideal, how did he come upon a formula for something equivalent to local cohomology—with maximal support—of the quotient by an arbitrary monomial ideal? Fairly easily: in computing the limit that defines the local cohomology $H_{I_{\Delta}}^i(S)$ supported on a squarefree monomial ideal I_{Δ} , Mustața was led to compute a simplicial formula for $\text{Ext}_S^i(S/I, S)$, where the monomial ideal I is a Frobenius power of I_{Δ} . But he also remarked that a similar formula holds for all monomial ideals I [Mus00, Section 2]. The Alexander duality observed in this section is a direct result of \mathbb{Z}^n -graded local duality (the ordinary kind, as opposed to with monomial support):

$$\begin{aligned} \text{Ext}^i(S/I, S)_{-\mathbf{a}-\mathbb{1}} &= \text{Ext}^i(S/I, S(-\mathbb{1}))(\mathbb{1})_{-\mathbf{a}-\mathbb{1}} \\ &= \text{Ext}^i(S/I, S(-\mathbb{1}))_{-\mathbf{a}} \\ &= (H_{\mathfrak{m}}^{n-i}(S/I)_{\mathbf{a}})^{\vee}. \end{aligned}$$

The formula for $\text{Ext}^i(S/I, S)$ is perhaps the simplest formula in this survey.

COROLLARY 6.5. *If $\mathbf{a} \in \mathbb{Z}^n$, then $\text{Ext}^i(S/I, S)_{\mathbf{a}} = \tilde{H}^{i-2}(\check{C}\Delta^{\mathbf{a}}; \mathbb{k})$.*

PROOF. The *Alexander duality isomorphism* (see [MiSt05, Theorem 5.6], for example) states that, for two simplicial complexes Δ and Δ^* that are Alexander dual inside of a simplex σ , there is a canonical isomorphism $\tilde{H}_{i-1}(\Delta; \mathbb{k}) = \tilde{H}^{|\sigma|-i-2}(\Delta^*; \mathbb{k})$. Now apply the Alexander duality in Theorem 6.3 to the simplicial formula in Theorem 3.8, using the local duality in Remark 6.4:

$$\begin{aligned} \text{Ext}^{n-i}(S/I, S)_{-\mathbf{a}-\mathbf{1}} &= (H_{\mathbf{m}}^i(S/I)_{\mathbf{a}})^{\vee} \\ &= \tilde{H}_{i-|\text{supp}(\mathbf{a}_-)|-1}(\check{C}\Delta_{\mathbf{a}}; \mathbb{k}) \\ &= \tilde{H}^{n-i-2}(\check{C}\Delta^{-\mathbf{a}-\mathbf{1}}; \mathbb{k}). \end{aligned}$$

Now, in the top and bottom lines, replace $n-i$ by i and \mathbf{a} by $-\mathbf{a}-\mathbf{1}$. (Avoid trying to do this in the middle line, which has a confusing $\text{supp}(\mathbf{a}_-)$ to deal with.) \square

REMARK 6.6. The formula in Corollary 6.5 seems to have been the first simplicial local cohomology—or equivalently, Ext —formula for arbitrary monomial ideals to appear in the literature [Mus00, Section 2], being a number of years earlier than the next, which seems to have been [Tak05, Theorem 1].

7. Koszul simplicial complexes

As we have seen, verifying the Cohen-Macaulay condition by analyzing the homology of a family of simplicial complexes, instead of a single simplicial complex as in the squarefree case, can be accomplished using a variety of simplicial complexes. The simplest family, however, is probably the following [MiSt05, Definition 5.9].

DEFINITION 7.1. Given a vector $\mathbf{b} \in \mathbb{N}^n$, set $\mathbf{b}' = \mathbf{b} - \text{supp}(\mathbf{b})$. For any monomial ideal I , the (*lower*) *Koszul simplicial complex* of S/I in degree $\mathbf{b} \in \mathbb{N}^n$ is

$$\text{K}\Delta_{\mathbf{b}} = \{\text{squarefree vectors } F \preceq \mathbf{b} \mid \mathbf{x}^{\mathbf{b}'+F} \notin I\}.$$

Equivalently, if $\Lambda_I \subseteq \mathbb{N}^n$ corresponds to the standard monomials (Definition 2.1),

$$\text{K}\Delta_{\mathbf{b}} = \{F \preceq \text{supp}(\mathbf{b}) \mid \mathbf{b}' + F \in \Lambda_I\}.$$

The motivation for the definition is analogous to the Čech simplicial complexes: just as the algebraic chain complex of $\check{C}\Delta_{\mathbf{a}}$ is the graded piece of the Čech complex of S/I in degree \mathbf{a} , the algebraic cochain complex of $\text{K}\Delta_{\mathbf{b}}$ is the graded piece of the Koszul complex of S/I in degree \mathbf{b} . In fact, the analogy between these families of simplicial complexes is even closer.

LEMMA 7.2. *For $\mathbf{b} \in \mathbb{N}^n$ and $\mathbf{a} = \mathbf{b} - \mathbf{1}$, the Koszul simplicial complex at \mathbf{b} equals*

$$\text{K}\Delta_{\mathbf{b}} = \{F - \text{supp}(\mathbf{a}_-) \mid (S/I)_{\mathbf{a}+F} = \mathbb{k}\}.$$

Consequently, $\check{C}\Delta_{\mathbf{a}}$ is a subcomplex of $\text{K}\Delta_{\mathbf{b}}$.

PROOF. The displayed formula for $\text{K}\Delta_{\mathbf{b}}$ follows immediately from Definition 7.1, given that $(S/I)_{\mathbf{a}+F}$ can only be nonzero if $\mathbf{a}+F$ has nonnegative coordinates, which occurs precisely when F contains $\text{supp}(\mathbf{a}_-)$. The containment $\check{C}\Delta_{\mathbf{a}} \subseteq \text{K}\Delta_{\mathbf{b}}$ follows from Definition 3.6, given that $(S_F/IS_F)_{\mathbf{a}}$ is nonzero precisely when $(S/I)_{\mathbf{a}+rF}$ is nonzero for all $r \geq 1$. \square

In certain special cases, as we shall see in Corollary 7.8, below, the inclusion $\check{C}\Delta_{\mathbf{a}} \subseteq K\Delta_{\mathbf{a}+\mathbf{1}}$ induces an isomorphism on a particular cohomology group, and this cohomology is precisely what is needed to detect the Cohen–Macaulay condition, or its failure. In general, a graded S -module M is Cohen–Macaulay precisely when its minimal free resolution of M has length equal to the codimension of M [MiSt05, Theorem 13.37.2]. Equivalently, the Betti numbers $\beta_{i,\mathbf{a}}(M)$ must vanish in homological degrees i greater than the codimension of M , for all \mathbf{a} . Homologically, the Betti number in degree \mathbf{a} and homological degree i is the \mathbb{k} -vector space dimension

$$\beta_{i,\mathbf{a}}(M) = \dim_{\mathbb{k}} \operatorname{Tor}_S^i(\mathbb{k}, M)_{\mathbf{a}}$$

[MiSt05, Lemma 1.32] (see also [MiSt05, Definition 8.22]). Therefore, the following generalization of Hochster’s formula for $\operatorname{Tor}_S^i(\mathbb{k}, S/I_{\Delta})_{\mathbf{a}}$ to arbitrary monomial quotients S/I [MiSt05, Theorem 5.11] yields the desired simplicial characterization of the Cohen–Macaulay condition in Corollary 7.4.

THEOREM 7.3. *Given $\mathbf{b} \in \mathbb{N}^n$ with support $F = \operatorname{supp}(\mathbf{b})$, a simplicial formula for the Betti number of S/I in degree \mathbf{b} is given by the \mathbb{k} -vector space dimension of*

$$\operatorname{Tor}_i(S/I, \mathbb{k})_{\mathbf{b}} = \tilde{H}^{|F|-i-1}(K\Delta_{\mathbf{b}}; \mathbb{k}).$$

PROOF. As mentioned after Definition 7.1, this can be proved by direct Koszul complex calculation. (The proof of [MiSt05, Theorem 5.11] actually proceeds via Alexander duality; see Remark 7.10.) \square

COROLLARY 7.4. *S/I is Cohen–Macaulay if and only if $\tilde{H}^{|F|-i-1}(K\Delta_{\mathbf{b}}; \mathbb{k}) = 0$ for all $\mathbf{b} \in \mathbb{N}^n$ and all $i > n - d$, where $d = \dim(S/I)$ and $F = \operatorname{supp}(\mathbf{b})$.*

For the record, Hochster’s Tor formula is the special case of Theorem 7.3 in which $I = I_{\Delta}$ is squarefree.

COROLLARY 7.5 (Hochster’s Tor formula). *The nonzero Betti numbers of S/I_{Δ} lie only in squarefree degrees F , and we have*

$$\operatorname{Tor}_i(S/I_{\Delta}, \mathbb{k})_F = \tilde{H}^{|F|-i-1}(\Delta|_F; \mathbb{k}),$$

where Δ_F is the subcomplex of all faces of Δ whose vertices lie in F .

PROOF. This was originally proved in [Hoc77]. It follows immediately from Theorem 7.3 and Definition 7.1. For additional information and context, this result is also [Sta96, Theorem II.4.8], [BrHe93, Remark 5.5.5], and [MiSt05, Corollary 5.12]. \square

REMARK 7.6. For general vectors $\mathbf{b} \in \mathbb{N}^n$ and $\mathbf{a} = \mathbf{b} - \mathbf{1}$, the relationship between the simplicial complexes $\check{C}\Delta_{\mathbf{a}}$ and $K\Delta_{\mathbf{b}}$ in Lemma 7.2 is mysterious, and perhaps difficult to analyze (if such an analysis is possible, it would be interesting to see). This is to be expected, since the Čech simplicial complexes compute local cohomology (Theorem 3.8), whereas the Koszul simplicial complexes compute Tor into \mathbb{k} (Theorem 7.3), which behaves quite differently from local cohomology, as a function of the \mathbb{Z}^n -graded degree.

Remark 7.6 notwithstanding, the socle of the first nonzero local cohomology precisely reflects the top Betti numbers. This holds quite generally for arbitrary finitely generated modules over regular local or graded rings. Here is the \mathbb{Z}^n -graded version. Its proof demonstrates that this result is a consequence of *duality for resolutions* [Mil00, Theorem 4.5] of \mathbb{Z}^n -graded modules. That result was developed in the context of Alexander duality for arbitrary (that is, not necessarily squarefree)

monomial ideals and modules, in response to the Eagon–Reiner theorem [ER98] and its generalizations. The point is that a minimal \mathbb{Z}^n -graded injective resolution of a finitely generated module M contains data equivalent to a minimal injective resolution of its Alexander dual, and therefore also enough data to recover minimal free resolutions of both M and its Alexander dual. These considerations include the fact—key in the proof of Theorem 7.7—that a minimal free resolution of M is equivalent to the maximal-support part of a minimal injective resolution of M .

THEOREM 7.7. *Let M be finitely generated and \mathbb{Z}^n -graded of projective dimension p ($\beta_{p,\mathbf{b}}(M) \neq 0$ for some $\mathbf{b} \in \mathbb{Z}^n$, and $\beta_{i,\mathbf{b}}(M) = 0$ for $i > p$). There is a natural map*

$$\mathrm{Tor}_i(M, \mathbb{k})_{\mathbf{b}} \rightarrow H_{\mathfrak{m}}^{n-i}(M)_{\mathbf{b}-\mathbf{1}}$$

for all i and all $\mathbf{b} \in \mathbb{Z}^n$. If $i = p$, and \mathbf{b} is any degree such that the Betti number $\beta_{p,\mathbf{b}}(M)$ is nonzero, then the natural map is an isomorphism:

$$\mathrm{Tor}_p(M, \mathbb{k})_{\mathbf{b}} \cong H_{\mathfrak{m}}^{n-p}(M)_{\mathbf{b}-\mathbf{1}} \text{ if } \beta_{p,\mathbf{b}} \neq 0.$$

PROOF. Let F_{\bullet} be a minimal free resolution of M . Then $\mathrm{Tor}_i(M, \mathbb{k})$ is the homology $H_i(F_{\bullet} \otimes \mathbb{k})$. In contrast, by [Mil00, Theorem 4.5.5], the \mathbb{Z}^n -graded translate $H_{\mathfrak{m}}^{n-i}(M)(-\mathbf{1})$ of the local cohomology up by $\mathbf{1}$ is the homology $H_i(F_{\bullet} \otimes E_{\mathbb{k}})$, where $E_{\mathbb{k}}$ is the \mathbb{Z}^n -graded injective hull of \mathbb{k} , also known as the Matlis dual of S . The existence of the natural map follows simply from the canonical inclusion of \mathbb{k} as the socle of $E_{\mathbb{k}}$. The natural map is an isomorphism when $i = p$ is maximal because, by minimality of the resolution, the differentials of $F_{\bullet} \otimes E_{\mathbb{k}}$ are zero on the socle. \square

As a consequence, we find that the mysterious inclusion from Lemma 7.2 has at least one tractable feature. Conveniently, this feature pertains precisely to the cohomology that reflects the Cohen–Macaulay property for S/I (Corollary 7.4).

COROLLARY 7.8. *Let S/I have projective dimension p . Assume that $\beta_{p,\mathbf{b}}(S/I) \neq 0$. Writing $\mathbf{a} = \mathbf{b} - \mathbf{1}$ and $F = \mathrm{supp}(\mathbf{b})$, the inclusion $\check{C}\Delta_{\mathbf{a}} \subseteq K\Delta_{\mathbf{b}}$ induces an isomorphism $\tilde{H}^{|F|-p-1}(\check{C}\Delta_{\mathbf{a}}; \mathbb{k}) = \tilde{H}^{|F|-p-1}(K\Delta_{\mathbf{b}}; \mathbb{k})$ on cohomology. \square*

REMARK 7.9. It is plausible that the Čech and lower Koszul simplicial complexes in Corollary 7.8 could be equal, but neither a proof nor a counterexample is known. It is also possible that additional hypotheses on S/I might be required; starting with the case where S/I is Cohen–Macaulay is probably a good idea.

REMARK 7.10. The formula in Theorem 7.3 comes in two flavors [MiSt05, Theorems 1.34 and 5.11] that are dual by definition, the latter derived as an immediate consequence of the former, by Alexander duality. Hence it would be possible—but more or less trivial—to present dual versions of the results in this section. In contrast, the space devoted to the Čech simplicial complexes $\check{C}\Delta_{\mathbf{a}}$ and their duals $\check{C}\Delta^{\mathbf{a}}$ in earlier sections was worthwhile: they arose separately, and their duality is neither by definition nor self-evident, being a manifestation of local duality (Remark 6.4).

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