Fourier Series with General Period

1 The Domain $[-p, p]$

The key idea here is that you can compute the Fourier coefficients for a function using any period of the sin and cos functions. To derive the coefficients for a function $f(x)$ represented by a Fourier series

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\frac{\pi}{p}x) + \sum_{k=1}^{\infty} b_k \sin(k\frac{\pi}{p}x) \tag{1}$$

defined on an interval $[-p, p]$ we use relations will gotten by adjusting the period of the sin and cos by multiplying x by $\frac{\pi}{p}$. These are

$$\int_{-p}^{p} \cos(\frac{\pi}{p}x)dw = 0 \qquad \int_{-p}^{p} \sin(\frac{\pi}{p}x)dx = 0$$

$$\int_{-p}^{p} \cos(m\frac{\pi}{p}x)\cos(n\frac{\pi}{p}x)dx = \begin{cases} p & \text{m=n} \\ 0 & \text{m} \neq \text{n} \end{cases} \qquad \int_{-p}^{p} \sin(m\frac{\pi}{p}x)\sin(n\frac{\pi}{p}x)dx = \begin{cases} p & \text{m=n} \\ 0 & \text{m} \neq \text{n} \end{cases} \tag{2}$$

$$\int_{-p}^{p} \cos(m\frac{\pi}{p}x)\sin(n\frac{\pi}{p}x)dx = 0 \text{ for any } m \neq n. \tag{3}$$

We can then compute the integrals

$$\begin{aligned}
\int_{-p}^{p} f(x)dx &= \int_{-p}^{p} (a_0 + \sum_{j=1}^{\infty} a_j \cos(j\frac{\pi}{p}x) + \sum_{j=1}^{\infty} b_j \sin(j\frac{\pi}{p}x))dx = 2pa_0 \\
\int_{-p}^{p} f(x)\cos(k\frac{\pi}{p}x)dx &= \int_{-p}^{p} (a_0 \cos(k\frac{\pi}{p}x) + \sum_{j=1}^{\infty} a_j \cos(j\frac{\pi}{p}x)\cos(k\frac{\pi}{p}x) + \sum_{j=1}^{\infty} b_j \sin(j\frac{\pi}{p}x)\cos(k\frac{\pi}{p}x))dx = pa_k \\
\int_{-p}^{p} f(x)\sin(k\frac{\pi}{p}x)dx &= \int_{-p}^{p} (a_0 \sin(k\frac{\pi}{p}x) + \sum_{j=1}^{\infty} a_j \cos(j\frac{\pi}{p}x)\sin(k\frac{\pi}{p}x) + \sum_{j=1}^{\infty} b_j \sin(j\frac{\pi}{p}x))\sin(k\frac{\pi}{p}x)dx = pb_k.
\end{aligned} \tag{4}$$

which leads to the same identities as in the period 2π case only dilated to a period $2p$ domain. So the coefficients of the Fourier series for a function f(x) of period $2p$ defined on the interval $[-p, p]$ are

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f(x)dx \tag{5}$$

$$a_k = \frac{1}{p}\int_{-p}^{p} f(x)\cos(k\frac{\pi}{p}x)dx \qquad b_k = \frac{1}{p}\int_{-p}^{p} f(x)\sin(k\frac{\pi}{p}x) \tag{6}$$

2 The Domain $[0, p]$

A similar argument will compute the identities for the coefficients of a Fourier series for a function defined on the interval $[0, p]$ of period p. Using

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\frac{2\pi}{p}x) + \sum_{k=1}^{\infty} b_k \sin(k\frac{2\pi}{p}x) \tag{7}$$

defined on the interval $[0, p]$ instead we will have that by adjusting the period of the sin and cos with the period shifting term $\frac{2\pi}{p}$,

$$\int_0^p \cos(\frac{2\pi}{p}x)dw = 0 \qquad \int_0^p \sin(\frac{2\pi}{p}x)dx = 0$$
$$\int_0^p \cos(m\frac{2\pi}{p}x)\cos(n\frac{2\pi}{p}x)dx = \begin{cases} \frac{p}{2} & \text{m=n} \\ 0 & \text{m} \neq \text{n} \end{cases} \qquad \int_0^p \sin(m\frac{2\pi}{p}x)\sin(n\frac{2\pi}{p}x)dx = \begin{cases} \frac{p}{2} & \text{m=n} \\ 0 & \text{m} \neq \text{n} \end{cases} \tag{8}$$

$$\int_0^p \cos(m\frac{2\pi}{p}x)\sin(n\frac{2\pi}{p}x)dx = 0 \, \text{for any } m \neq n. \tag{9}$$

and so in the integrals we compute that

$$\int_0^p f(x)dx = \int_0^p (a_0 + \sum_{j=1}^{\infty} a_j \cos(j\frac{2\pi}{p}x) + \sum_{j=1}^{\infty} b_j \sin(j\frac{2\pi}{p}x))dx = pa_0$$
$$\int_0^p f(x)\cos(k\frac{2\pi}{p}x)dx = \int_0^p (a_0\cos(k\frac{2\pi}{p}x) + \sum_{j=1}^{\infty} a_j \cos(j\frac{2\pi}{p}x)\cos(k\frac{2\pi}{p}x) + \sum_{j=1}^{\infty} b_j \sin(j\frac{2\pi}{p}x)\cos(k\frac{2\pi}{p}x))dx = \frac{p}{2}a_k$$
$$\int_0^p f(x)\sin(k\frac{2\pi}{p}x)dx = \int_0^p (a_0\sin(k\frac{2\pi}{p}x) + \sum_{j=1}^{\infty} a_j \cos(j\frac{2\pi}{p}x)\sin(k\frac{2\pi}{p}x) + \sum_{j=1}^{\infty} b_j \sin(j\frac{2\pi}{p}x))\sin(k\frac{2\pi}{p}x)dx = \frac{p}{2}b_k. \tag{10}$$

which leads to the following identities for the coefficients of the Fourier series of a function f(x) of period p defined on the interval $[0, p]$

$$a_0 = \frac{1}{p}\int_0^p f(x)dx \tag{11}$$

$$a_k = \frac{2}{p}\int_0^p f(x)\cos(k\frac{2\pi}{p}x)dx \qquad b_k = \frac{2}{p}\int_0^p f(x)\sin(k\frac{2\pi}{p}x) \tag{12}$$

You should experiment with different values of p to get a feeling for how these general cases reduce to the cases you have seen already, e.g. $p = \{1, 2, \pi, 2\pi\}$.