Abstract. Let $K$ be a quadratic number field, and let $\zeta_K(s)$ denote the Dedekind zeta-function attached to $K$. Using the mixed second moments of derivatives of $\zeta_K(\frac{1}{2}+it)$, we prove the existence of gaps between consecutive zeros of $\zeta_K(s)$ on the critical line which are at least $\sqrt{6} = 2.44949 \ldots$ times the average spacing.

Let $K$ be a number field and $\mathcal{O}_K$ its ring of integers. The Dedekind zeta-function attached to $K$ is defined by

$$\zeta_K(s) = \sum_{a \subset \mathcal{O}_K} \frac{1}{N(a)^s} = \prod_{p \subset \mathcal{O}_K} \left(1 - \frac{1}{N(p)^s}\right)^{-1}, \quad \Re(s) > 1,$$

where $a$ and $p$ run over the nonzero ideals and prime ideals of $\mathcal{O}_K$, respectively. Let $K$ be a quadratic field with discriminant $d$, and let $\chi_d$ be the Kronecker symbol of $d$. Then the Dedekind zeta-function factors as

$$\zeta_K(s) = \zeta(s)L(s,\chi_d),$$

where $\zeta(s)$ is the Riemann zeta-function and $L(s,\chi_d)$ is the Dirichlet $L$-function associated to $\chi_d$.

This note studies the vertical distribution of the zeros of $\zeta_K(s)$ in the critical strip, which we denote by $\rho_K = \beta + i\gamma$. It has been shown that for an imaginary quadratic number field $K$, the vertical distribution of the nontrivial zeros of $\zeta_K(s)$ is related to the existence or non-existence of Landau-Siegel zeros and hence the size of the class number of $K$. This correspondence is described in the work of Conrey and Iwaniec [10]; see also Montgomery and Weinberger [22]. This circle of ideas is often referred to as the Deuring-Heilbronn phenomenon. For a very nice overview of the Deuring-Heilbronn phenomenon and its implications, see Stopple’s survey article [27].

For a real or imaginary quadratic number field of discriminant $d$, it is known [19, Theorem 5.31] that for $T \geq 2$, we have

$$N_K(T) := \sum_{0 < \gamma \leq T} 1 = \frac{T}{\pi} \log \frac{\sqrt{|d|}T}{(2\pi e)^2} + O\left(\log(\sqrt{|d|}T)\right).$$

Consider the sequence $0 < \gamma_1 \leq \gamma_2 \leq \ldots$ of consecutive ordinates of the nontrivial zeros of $\zeta_K(s)$, and note that the average size of $\gamma_{n+1} - \gamma_n$ is $\pi/\log(\sqrt{|d|}\gamma_n)$. Normalizing, let

$$\mu_K := \liminf_{n \to \infty} \frac{\gamma_{n+1} - \gamma_n}{\pi/\log(\sqrt{|d|}\gamma_n)}$$

and

$$\lambda_K := \limsup_{n \to \infty} \frac{\gamma_{n+1} - \gamma_n}{\pi/\log(\sqrt{|d|}\gamma_n)}.$$

By definition we have $\mu_K \leq 1 \leq \lambda_K$, however it is conjectured that $\mu_K = 0$ and $\lambda_K = \infty$. In other words, we expect that there are arbitrarily small and large normalized gaps between consecutive nontrivial zeros

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of Dedekind zeta-functions of quadratic number fields. While we expect $\mu_K = 0$, this is not due to the presumption of coincident nontrivial zeros of $\zeta(s)$ and $L(s, \chi_d)$. On the contrary, we expect that the zeros of $\zeta_K(s)$ are simple. Conrey, Ghosh, and Gonek [7] have shown that the number of simple zeros of $\zeta_K(s)$ with $0 < \gamma \leq T$ exceeds $T^{6/11}$ for sufficiently large $T$. In [8], the same authors show, assuming the generalized Riemann hypothesis for Dirichlet $L$-functions, that a positive proportion of the zeros of $\zeta_K(s)$ are simple. In general, it is conjectured that any two distinct primitive $L$-functions should have no shared zero.

That $\mu_K < 1 < \lambda_K$ is an open question, and in particular there do not seem to be any quantitative results concerning the sizes of $\mu_K$ or $\lambda_K$. This is in contrast to the distribution of the zeros of the Riemann zeta-function, where there is an abundance of results, both unconditional and assuming various unproved hypotheses. See, for instance, [1], [2], [3], [4], [5], [6], [9], [11], [12], [13], [14], [20], [21], [24], [26], and [29].

The object of this note is to provide a nontrivial lower bound for $\lambda_K$. Towards this goal, we prove the following unconditional theorem.

**Theorem 1.** Let $T \geq 2$ and $\varepsilon > 0$. Let $K$ be a quadratic number field of discriminant $d$ with $|d| \leq T^{5-\varepsilon}$. There exists a subinterval of $[T, 2T]$ having length at least

$$\sqrt{6} \cdot \frac{\pi}{\log \sqrt{|d|} \cdot T} \left(1 + O(d^\varepsilon \log^{-1} T)\right)$$

for which the function $t \mapsto \zeta_K(1/2 + it)$ is free of zeros.

Theorem 1 does not, *a fortiori*, state anything about the quantity $\lambda_K$. However, if we assume the generalized Riemann hypothesis for $\zeta_K(s)$, then Theorem 1 immediately implies the following inequality for $\lambda_K$.

**Corollary 2.** Assume the generalized Riemann hypothesis for $\zeta_K(s)$. Then $\lambda_K \geq \sqrt{6}$. In particular, there are infinitely many normalized gaps between consecutive zeros of $\zeta_K(s)$ which are greater than $\sqrt{6} - \varepsilon$ times the average spacing for any $\varepsilon > 0$.

The constant $\sqrt{6}$ in Corollary 2 is larger than one might expect since the same method of proof applied to the Riemann zeta-function only exhibits gaps between nontrivial zeros of $\zeta(s)$ of size $\sqrt{3}$ times the average spacing. (See [13].) Moreover, in contrast to Theorem 1 and its corollary, establishing a nontrivial upper bound on $\mu_K$ seems to be more difficult due to the connection to the Deuring-Heilbronn phenomenon and the class number problem for imaginary quadratic fields mentioned above.

We prove Theorem 1 by combining the mixed second moments of derivatives of $\zeta_K(s)$ and an argument of R. R. Hall [13]. In 1926, Ingham [18] proved that for $s = 1/2 + it$ and $|\alpha|, |\beta| < 1/2$, we have

$$\int_0^T \zeta(s + \alpha)\zeta(1-s + \beta) dt = \int_0^T \left(\zeta(1+\alpha+\beta) + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \zeta(1-\alpha-\beta)\right) \left(1 + O(t^{-1/2})\right) dt.$$

This ‘shifted’ moment reveals a beautiful underlying structure which allows one to deduce lower order terms and moments of derivatives of $\zeta(s)$ via differentiation and Cauchy’s integral formula. For instance, Ingham’s theorem can be used to show that, for fixed $\mu, \nu \in \mathbb{N}$,

$$\int_0^T \zeta^{(\mu)}(1/2 + it)\zeta^{(\nu)}(1/2 - it) dt = \frac{(-1)^{\mu+\nu}}{\mu+\nu+1} T (\log T)^{\mu+\nu+1} + O(T (\log T)^{\mu+\nu}),$$

where $\zeta^{(\mu)}(s)$ denotes the $\mu$th derivative of $\zeta(s)$. We make use of a similar shifted moment result for a Dedekind zeta-function of a quadratic number field due to Heap [16] to obtain the mixed second moments of derivatives of $\zeta_K(s)$ on the critical line. In particular, the proof of Theorem 1 requires asymptotic estimates
of the mixed second moments of \( \zeta_K(\frac{1}{2}+it) \) and \( \zeta'_K(\frac{1}{2}+it) \) with a uniform error. We obtain these by way of the following theorem.

**Theorem 3.** Let \( K \) be the quadratic number field with discriminant \( d \). Let \( T \geq 2 \), and \( \mu, \nu \) be non-negative integers. We have

\[
\int_T^{2T} \zeta_K^{(\mu)}(\frac{1}{2}+it)\zeta_K^{(\nu)}(\frac{1}{2}+it)\,dt = \frac{(\mu+\nu)(\mu+\nu+1)}{(\mu+\nu+2)(\mu+\nu+1)}2C_dT(\log T)^{\mu+\nu+2} + O(\mu!\nu!d^\epsilon C_dT(\log T)^{\mu+\nu+1}),
\]

where the constant

\[
C_d := \frac{6}{\pi^2} \prod_{p|d} \left( 1 + \frac{1}{p} \right)^{-1} L^2(1, \chi_d).
\]

Special cases of Theorem 3 are known by the work of Motohashi [23] and Weinstein [28], however we require the more general case to prove Theorem 1. We deduce Theorem 3 from the following recent result of Heap [16].

**Theorem 4.** (Heap) Let \( K \) be the quadratic number field with discriminant \( d \). Let \( s = 1/2+it \) and \( \alpha, \beta \in \mathbb{C} \) such that \( |\alpha|, |\beta| \ll 1/\log(\sqrt{d}T) \). Then we have

\[
\int_T^{2T} \zeta_K(s+\alpha)\zeta_K(1-s+\beta)\,dt
= \int_T^{2T} \left\{ \prod_p \left( 1 - \frac{1}{p^{1+\alpha+\beta}} \right) \prod_{p|d} \left( 1 + \frac{1}{p^{1+\alpha+\beta}} \right)^{-1} \zeta_K^2(1+\alpha+\beta)
\right.
\]
\[
+ \left( \frac{t}{2\pi} \right)^{-\alpha-\beta} \frac{6}{\pi^2} \prod_{p|d} \left( 1 - \frac{1}{p^2} \right)^{-1} \prod_{p|d} \left( 1 - \frac{1}{p^{1+\alpha+\beta}} \right) L^2(1, \chi_d)\zeta(1+\alpha+\beta)\zeta(1-\alpha-\beta)
\]
\[
+ \frac{1}{d^{\alpha+\beta}} \left( \frac{t}{2\pi} \right)^{-\alpha-\beta} \frac{6}{\pi^2} \prod_{p|d} \left( 1 - \frac{1}{p^2} \right)^{-1} \prod_{p|d} \left( 1 - \frac{1}{p^{1-\alpha-\beta}} \right) L^2(1, \chi_d)\zeta(1+\alpha+\beta)\zeta(1-\alpha-\beta)
\]
\[
+ \frac{1}{d^{\alpha+\beta}} \left( \frac{t}{2\pi} \right)^{-2\alpha-2\beta} \prod_p \left( 1 - \frac{1}{p^{2-2\alpha-2\beta}} \right) \prod_{p|d} \left( 1 + \frac{1}{p^{1-\alpha-\beta}} \right)^{-1} \zeta_K^2(1-\alpha-\beta)
\}
\]

\[(3) \]

where the constant \( C_d \) is defined in (2).

**Proof.** This is a consequence of [16, Theorem 1], letting \( h = k = 1 \). □

Prior to the work of Heap [16], the author independently derived Theorem 4 using a method of Ramachandran [25]. Using different techniques, Heap computes the second moment of a Dedekind zeta-function of a quadratic field times an arbitrary Dirichlet polynomial of length \( T^{1/11-\epsilon} \).

**Proof of Theorem 3.** Let \( \epsilon > 0 \) be an arbitrary constant, \( s = 1/2+it \), and \( T \geq 2 \) be fixed. We first simplify the integral on the right-hand side of (3) by considering each factor of each term of the integrand. Since \( \alpha+\beta \ll 1/\log(\sqrt{d}T) \), it follows that \( d^{-\alpha-\beta} = 1 + O((\alpha+\beta)d^\epsilon) \). The Euler products on the right-hand side of (3) can be simplified as

\[
\prod_p \left( 1 - \frac{1}{p^{2\alpha+\beta}} \right) = \prod_p \left( 1 - \frac{1}{p^2} \right) \left( 1 + O((\alpha+\beta)d^\epsilon) \right) = \frac{6}{\pi^2} \left( 1 + O((\alpha+\beta)d^\epsilon) \right),
\]

3
\[
\prod_{p|d} \left(1 + \frac{1}{p^{1+\alpha+\beta}}\right)^{-1} = \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} \left(1 + O((\alpha+\beta)d^r)\right),
\]
and
\[
\prod_{p|d} \left(1 - \frac{1}{p^{1+\alpha+\beta}}\right) = \prod_{p|d} \left(1 - \frac{1}{p}\right) \left(1 + O((\alpha+\beta)d^r)\right).
\]

The factorization given in (1) implies that
\[
\zeta_K (1 \pm (\alpha+\beta)) = L(1, \chi_d)\zeta (1 \pm (\alpha+\beta)) \left(1 + O((\alpha+\beta)d^r)\right).
\]

Furthermore, since \( t \in [T, 2T] \), we have that \((t/2\pi)^{-\alpha-\beta} = T^{-\alpha-\beta}(1 + O(1/\log T))\). Using these estimates, we find that
\[
\int_T^{2T} \zeta_K(s+\alpha)\zeta_K(1-s+\beta) \, dt = \int_T^{2T} \left\{ \frac{6}{\pi^2} \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} L^2(1, \chi_d)\zeta^2(1+\alpha+\beta) \right\} \, dt
\]
\[
+ 2 \int_T^{2T} \left\{ \frac{6}{\pi^2} \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} L^2(1, \chi_d)\zeta(1+\alpha+\beta)\zeta(1-\alpha-\beta)T^{-\alpha-\beta} \right\} \, dt
\]
\[
+ \int_T^{2T} \left\{ \frac{6}{\pi^2} \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} L^2(1, \chi_d)\zeta^2(1-\alpha-\beta)T^{-2\alpha-2\beta} \right\} \, dt
\]
\[
+ O(d^r C_d T \log T)
\]
\[
:= I_1 + 2I_2 + I_3 + O(d^r C_d T \log T),
\]
say. Since \( \zeta(1-s) = 1/s + O(1) \), we can express the three integrals as
\[
I_1 = (\alpha+\beta)^{-2} C_d T + O(d^r C_d T \log T), \quad I_2 = - (\alpha+\beta)^{-2} C_d T^{-\alpha-\beta+1} + O(d^r C_d T \log T),
\]
and
\[
I_3 = (\alpha+\beta)^{-2} C_d T^{-2\alpha-2\beta+1} + O(d^r C_d T \log T).
\]

Finally, noting that
\[
T^{-\delta(\alpha+\beta)} = \sum_{n=0}^\infty \frac{(-1)^n \delta^n (\alpha+\beta)^n (\log T)^n}{n!},
\]
we simplify \( I_1 + 2I_2 + I_3 \) to conclude that
\[
\int_T^{2T} \zeta_K(s+\alpha)\zeta_K(1-s+\beta) \, dt = F(\alpha+\beta; T) + O(d^r C_d T \log T),
\]
where
\[
F(\alpha+\beta; T) := 2C_d T \sum_{n=0}^\infty \frac{(-1)^n (\alpha+\beta)^n (\log T)^{n+2}}{(n+2)!} \left(2^{n+1} - 1\right).
\]

We now follow an argument of Ingham [18] to complete the proof. Let
\[
R(\alpha, \beta; T) := \int_T^{2T} \zeta_K(s+\alpha)\zeta_K(1-s+\beta) \, dt - F(\alpha+\beta; T).
\]
Then \( R(\alpha, \beta; T) \) is an analytic function of two complex variables \( \alpha \) and \( \beta \) when \( \Re(\alpha), \Re(\beta) < 1/2, \) and
\[
R(\alpha, \beta; T) = O(d^r C_d T \log T)
\]
holds by Theorem 4. Differentiating (5), it follows that

\begin{equation}
\int_T^{2T} \zeta_K^{(\mu)}(s) \zeta_K^{(\nu)}(1-s) \, dt = \frac{\partial^{\mu+\nu} F(\alpha+\beta; T)}{\partial \alpha^\mu \partial \beta^\nu} + R_{\mu,\nu}(\alpha, \beta; T),
\end{equation}

where \( \mu \) and \( \nu \) are fixed nonnegative integers and

\[ R_{\mu,\nu}(\alpha, \beta; T) := \frac{\partial^{\mu+\nu} R(\alpha, \beta; T)}{\partial \alpha^\mu \partial \beta^\nu}. \]

Let \( \mathcal{C} = \{ w \in \mathbb{C}; |w - \alpha| = 1/\log T \} \). By the Cauchy integral formula and (6), we have

\[ \frac{\partial^\mu}{\partial \alpha^\mu} R(\alpha, \beta; T) = \frac{\mu!}{2\pi i} \int_{\mathcal{C}} \frac{R(w, \beta; T)}{(w - \alpha)^{\mu+1}} \, dw = O(\mu!d^\mu C_d \log T)^{\mu+1}. \]

Appealing to the Cauchy integral formula once more, we deduce that

\[ R_{\mu,\nu}(\alpha, \beta; T) := \frac{\partial^{\mu+\nu}}{\partial \alpha^\mu \partial \beta^\nu} R(\alpha, \beta; T) = O(\mu!\nu!d^{\mu+\nu} C_d (\log T)^{\mu+\nu+1}). \]

Thus (7), with \( \alpha = \beta = 0 \), gives

\begin{equation}
\int_0^T \zeta_K^{(\mu)}(\frac{1+i}{2}) \zeta_K^{(\nu)}(\frac{1-i}{2}) \, dt = \left[ \frac{\partial^{\mu+\nu} F(\alpha+\beta; T)}{\partial \alpha^\mu \partial \beta^\nu} \right]_{\alpha=0, \beta=0} + O(\mu!\nu!d^{\mu+\nu} C_d (\log T)^{\mu+\nu+1}),
\end{equation}

and it remains only to calculate the first term on the right-hand side. By differentiating (4) with respect to \( \alpha \) and \( \beta \) and simplifying, we determine that

\begin{equation}
\left[ \frac{\partial^{\mu+\nu} F(\alpha+\beta; T)}{\partial \alpha^\mu \partial \beta^\nu} \right]_{\alpha=0, \beta=0} = \frac{(-1)^{\mu+\nu}(2^{\mu+\nu+1}-1)}{(\mu+\nu+2)(\mu+\nu+1)} 2C_d (\log T)^{\mu+\nu+2}. \end{equation}

Theorem 3 now follows upon inserting (9) into (8).

We now demonstrate how to obtain the lower bound in Theorem 1. The proof is a variation of a method of R. R. Hall [13] using some ideas of Bredberg [1]. We begin by defining the function

\begin{equation}
f(t) := e^{ivt \log T} \zeta_K(\frac{1+i}{2} + it),
\end{equation}

where \( v \) is a real constant that will be chosen later. By Stirling’s formula, \( f(t) \) mimics the analogue of the Hardy Z-function for \( \zeta_K(s) \). Fix \( K \), and let \( \gamma \) denote an ordinate of a zero of \( \zeta_K(s) \) on the critical line \( \Re(s) = 1/2 \). Note that \( f(t) \) has the same zeros as \( \zeta_K(\frac{1+i}{2} + it) \), that is, \( f(t) = 0 \) if and only if \( t = \gamma \). Let \( \{ \gamma_1, \gamma_2, \ldots, \gamma_N \} \) denote the set of distinct zeros of \( f(t) \) in the interval \([T, 2T]\) arranged in non-decreasing order and ignoring multiplicity. Furthermore, let

\[ \kappa_T = \max \{ \gamma_{n+1} - \gamma_n : T+1 \leq \gamma_n \leq 2T-1 \}, \]

and note that \( \lambda_K \geq \limsup \kappa_T \). Without loss of generality, we may assume that

\begin{equation}
\gamma_1 - T \ll 1 \quad \text{and} \quad 2T - \gamma_N \ll 1,
\end{equation}

as otherwise there exist zeros \( \gamma_0 \leq \gamma_1 \) and \( \gamma_{N+1} \geq \gamma_N \) such that \( \gamma_0 - \gamma_1 \) and \( \gamma_{N+1} - \gamma_N \) are \( \gg 1 \), and

Theorem 1 holds for this reason. In order to obtain a lower bound on \( \kappa_T \), we require the following lemma.
Lemma 5. Let \( y : [a,b] \to \mathbb{C} \) be a continuously differentiable function and suppose that \( y(a) = y(b) = 0 \). Then
\[
\int_a^b |y(x)|^2 \, dx \leq \left( \frac{b-a}{\pi} \right)^2 \int_a^b |y'(x)|^2 \, dx.
\]

Proof. This is a variation of a well-known inequality of Wirtinger [15, Theorem 256] due to Bredberg [1, Corollary 1].

With this setup, we now prove Theorem 1.

Proof of Theorem 1. Let \( \varepsilon > 0 \) be a small positive constant which may vary from line to line, and let \( f(t) \) be the function defined in (10). By the definition of \( \kappa_T \), for each pair of consecutive zeros of \( f(t) \) in the interval \( [T, 2T] \), we have
\[
\int_{\gamma_n}^{\gamma_{n+1}} |f(t)|^2 \, dt \leq \frac{\kappa_T^2}{\pi^2} \int_{\gamma_n}^{\gamma_{n+1}} |f'(t)|^2 \, dt.
\]

Summing both sides of the equation in (12) over \( n \) for \( n = 1, 2, \ldots, N - 1 \), it follows that
\[
\int_{\gamma_1}^{\gamma_N} |f(t)|^2 \, dt \leq \frac{\kappa_T^2}{\pi^2} \int_{\gamma_1}^{\gamma_N} |f'(t)|^2 \, dt.
\]

By Weyl’s bound for the zeta-function, \( \zeta(\frac{1}{2}+it) \ll t^{\frac{1}{2}+\varepsilon} \), and the subconvexity bound \( L(\frac{1}{2}+it, \chi_d) \ll |td|^{\frac{1}{4}+\varepsilon} \) due to Heath-Brown [17], we see that \( |f(t)| \ll t^{\frac{1}{12}+\varepsilon}d^{\frac{1}{2}+\varepsilon} \) for \( T \leq t \leq 2T \) and \( \varepsilon > 0 \). Therefore, by the assumption in (11), we have
\[
\int_T^{2T} |f(t)|^2 \, dt \leq \frac{\kappa_T^2}{\pi^2} \int_T^{2T} |f'(t)|^2 \, dt + O(d^{\frac{3}{2}+\varepsilon}T^{\frac{17}{3}+\varepsilon}).
\]

Note that \( |f(t)|^2 = |\zeta_K(\frac{1}{2}+it)|^2 \) and
\[
|f'(t)|^2 = |\zeta'_{\kappa}(\frac{1}{2}+it)|^2 + v^2 \log^2 T|\zeta_{\kappa}(\frac{1}{2}+it)|^2 + 2v \log T \cdot \Re \left( \zeta'_{\kappa}(\frac{1}{2}+it)\zeta_{\kappa}(\frac{1}{2}+it) \right).
\]

Theorem 3 implies that
\[
\int_T^{2T} |\zeta_{\kappa}(\frac{1}{2}+it)|^2 \, dt = C_\delta T \log^2 T + O(d^{\varepsilon}C_\delta T \log T),
\]

\[
\int_T^{2T} \zeta'_{\kappa}(\frac{1}{2}+it)\zeta_{\kappa}(\frac{1}{2}+it) \, dt = -C_\delta T \log^3 T + O(d^{\varepsilon}C_\delta T \log^2 T),
\]

and
\[
\int_T^{2T} |\zeta'_{\kappa}(\frac{1}{2}+it)|^2 \, dt = \frac{7}{6} C_\delta T \log^4 T + O(d^{\varepsilon}C_\delta T \log^3 T),
\]

where \( C_\delta \) is the constant in (2). By combining the estimates in (13) – (17), we find that
\[
\frac{\kappa_T^2}{\pi^2} \geq \frac{6}{6v^2 - 12v + 7 \log^2 T} \left( 1 + O(d^{\varepsilon} \log^{-1} T) \right),
\]

uniformly for \( |d| \leq T^{\frac{1}{2}-\varepsilon} \). The choice of \( v = 1 \) minimizes \( 6v^2 - 12v + 7 \), the minimum value being 1. We conclude that
\[
\kappa_T \geq \frac{\sqrt{6\pi}}{\log(\sqrt{|d|}T)} \left( 1 + O(d^{\varepsilon} \log^{-1} T) \right).
\]

This completes the proof of Theorem 1. \(\square\)
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