§A. The equation $d\omega = F(\omega)$.

Let M^3 be a smooth 3-manifold and let $F\colon \Lambda^1(T^*M) \to \Lambda^2(T^*M)$ be a smooth fiber-preserving map (F is not assumed to be linear). I want to consider the differential equation $d\omega = F(\omega)$ for a 1-form ω on M^3. Since Λ^1 and Λ^2 have the same rank, this is a determined first-order system of equations for ω.

However, even though this is a determined system, it is easy to see that there exist maps F for which no solution ω exists. A simple example is got by letting F be constant on each fiber with $F(T^*_m) = \varphi_m \in \Lambda^2(T^*_m)$ for each $m \in M$. Then the equation to be solved becomes $d\omega = \varphi$, which has no solutions (even locally) if $d\varphi \neq 0$. This example is not typical. The discussion which follows will show that, for a "generic" F (at least in the real-analytic category), the equation $d\omega = F(\omega)$ does possess local solutions. Presumably, this is also true in the smooth category once the notion of "generic" has been made sufficiently precise.

To local coordinates x^1, x^2, x^3 on $U \subset M$, there are associated canonical coordinates p_1, p_2, p_3 on T^*U so that $p_i(\omega)$ is the dx^i component of ω for $1 \leq i \leq 3$. Then $F_{|U}\colon T^*U \to \Lambda^2(T^*U)$ can be written in the form

$$F(p_1 dx^1 + p_2 dx^2 + p_3 dx^3) = F^1(x,p)\, dx^2 \wedge dx^3 + F^2(x,p)\, dx^3 \wedge dx^1 + F^3(x,p)\, dx^1 \wedge dx^2,$$

where the F^i are smooth functions of $\{x^i, p_j\}_{1 \leq i,j \leq 3}$. The equation to be solved may now regarded as an equation for the p_j as functions of the x^i and written in the form

$$dp_1 \wedge dx^1 + dp_2 \wedge dx^2 + dp_3 \wedge dx^3 = F^1\, dx^2 \wedge dx^3 + F^2\, dx^3 \wedge dx^1 + F^3\, dx^1 \wedge dx^2.$$

Now, if $\omega = p_1\, dx^i$ satisfies $d\omega = F(\omega)$, then $d\big(F(\omega)\big) = d\big(d\omega\big) = 0$ or, more explicitly,

$$dF^1 \wedge dx^2 \wedge dx^3 + dF^2 \wedge dx^3 \wedge dx^1 + dF^3 \wedge dx^1 \wedge dx^2 = 0.$$

The vanishing of this latter 3-form represents yet a fourth first-order equation which the p_i must satisfy. This extra equations arises because, as a differential operator, the exterior derivative $d\colon \Omega^1 \to \Omega^2$ is degenerate. Its symbol $\sigma_\xi(d)\colon \Lambda^1 \to \Lambda^2$ is given by $\sigma_\xi(d)\big(\eta\big) = \xi\wedge\eta$ and hence is never injective for any $\xi \in T^*$. (In other words, every co-direction is characteristic.)

Of course, one can write the above four equations in the form of PDE:

$$\begin{aligned} \frac{\partial p_i}{\partial x^j} - \frac{\partial p_j}{\partial x^i} &= -F^k(x,p) \\ \sum_{i,j} \frac{\partial F^i}{\partial p_j}(x,p) \frac{\partial p_j}{\partial x^i} &= -\sum_i \frac{\partial F^i}{\partial x^i}(x,p) \end{aligned}$$

where, in the first three equations, (i,j,k) is an even permutation of $(1,2,3)$. Let

$$Q^{ij}(x,p) = \frac{1}{2}\left(\frac{\partial F^i}{\partial p_j}(x,p) + \frac{\partial F^j}{\partial p_i}(x,p)\right).$$

Then the reader can easily check that a co-vector $\xi = \xi_i dx^i$ is characteristic at (x,p) for this system of four equations (in the classical PDE sense of "characteristic") if and only if $Q^{ij}(x,p)\xi_i\xi_j = 0$.

The quadratic form Q^{ij} has an intrinsic formulation which will be important in later discussions, so I will describe it here. Let $X^6 = T^*M$ and let $\pi\colon X \to M$ denote the projection. For $\alpha \in X$, with $\pi(\alpha) = m$ (i.e., $\alpha \in T^*_m M$), there is a canonical isomorphism of $\ker\big(\pi'(\alpha)\big)$ (i.e., the "vertical tangent space at α ") with $T^*_m M$. Since F maps T^*_m to $\Lambda^2(T^*_m) \simeq T_m \otimes \Lambda^3(T^*_m)$, the differential of F at α restricts to $\ker(\pi'(\alpha) \simeq T^*_m$ to give a well-defined map $D_F(\alpha)\colon T^*_m \to \Lambda^2(T^*_m) \simeq T_m \otimes \Lambda^3(T^*_m)$. By transpose, one may regard $D_F(\alpha)$ as an element of $T_m \otimes T_m \otimes \Lambda^3(T^*_m)$. Since there is a canonical decomposition $T_m \otimes T_m \simeq S^2(T_m) \oplus \Lambda^2(T_m)$, $DF(\alpha)$ splits into the sum of an element $Q_F(\alpha)$ in $S^2(T_m) \otimes \Lambda^3(T^*_m)$ and an element $A_F(\alpha)$ in $\Lambda^2(T_m) \otimes \Lambda^3(T^*_m) \simeq T^*_m$. In local coordinates, writing $\alpha = p_i\, dx^i$, there are the explicit expressions

$$Q_F(x,p) = Q^{ij}(x,p)\, \frac{\partial}{\partial x^i} \circ \frac{\partial}{\partial x^j}\ \otimes dx^1 \wedge dx^2 \wedge dx^3$$

and

$$A_F(x,p) = A_k(x,p)\,dx^k = \frac{1}{2}\left(\frac{\partial F^i}{\partial p_j}(x,p) - \frac{\partial F^j}{\partial p_i}(x,p)\right) dx^k$$

where the latter expression is to be summed over the even permutations (i,j,k) of $(1,2,3)$. Thus, Q_F is a section of $S^2(L^*) \otimes \Lambda^3(L)$ and A_F is a section of $L = \pi^*(T^*M)$.

Theorem 1: *Suppose that F is real-analytic in some local coordinate system containing $\alpha \in X = T^*M$ and suppose that $Q_F(\alpha) \neq 0$. Then on a neighborhood U of $m = \pi(\alpha)$, there exists $\omega \in \Omega^1(U)$ satisfying $\omega_m = \alpha$ and $d\omega = F(\omega)$ on U.*

PROOF: I will first construct an appropriate differential system on X. It is easy to see that there exists a unique 2-form Θ on X which is given, in local canonical coordinates (x^i, p_j), by the formula

$$\Theta = \sum_i dp_i \wedge dx^i - (F^1\,dx^2 \wedge dx^3 + F^2\,dx^3 \wedge dx^1 + F^3\,dx^1 \wedge dx^2)$$

(When $F \equiv 0$, then Θ is just the canonical symplectic form on $X = T^*M$.) In these coordinates,

$$d\Theta = -(dF^1 \wedge dx^2 \wedge dx^3 + dF^2 \wedge dx^3 \wedge dx^1 + dF^3 \wedge dx^1 \wedge dx^2).$$

If ω is any 1-form on M regarded as a section of $X \to M$, then Θ has the "reproducing" property

$$\omega^*(\Theta) = d\omega - F(\omega)$$
$$\omega^*(d\Theta) = -d\big(F(\omega)\big).$$

Let $\mathcal{I}$ be the differential system on X generated by $\{\Theta, d\Theta\}$. The (local) solutions of $d\omega = F(\omega)$ are exactly the (local) sections $\omega\colon U \to X_U$ which are integrals of $\mathcal{I}$, i.e., which satisfy $\omega^*(\mathcal{I}) = 0$.

Let $L \subset T^*X$ be the bundle of π-semi-basic 1-forms. In local (x,p)-coordinates, L is spanned by $\{dx^i \mid i = 1,2,3\}$. The pair $(\mathcal{I}, L)$ is in linear form since

$$d\Theta = \sum\sum_m \frac{\partial F^i}{\partial p_m} dp_m \wedge dx^j \wedge dx^k + E(x,p)\,dx^1 \wedge dx^2 \wedge dx^3$$

where the outer sum is over the even permutations of (1,2,3) and $E = \sum_i \partial F^i/\partial x^i$. Using the definitions of Q^{ij} and A_k given above, a little index juggling gives

$$d\Theta = \sum\sum_m Q^{im} dp_m \wedge dx^j \wedge dx^k - (A_F) \wedge \Theta + \tilde{E}\,dx^1 \wedge dx^2 \wedge dx^3$$

where $\tilde{E} = E - \sum_k A_k F^k$ is a function of (x,p). Thus, $\mathcal{I}$ is generated by the forms $\{\Theta,\ d\Theta + (A_F){\wedge}\Theta\}$ which in local coordinates, take the forms

$$\Theta = \sum_k dp_k \wedge dx^k - \sum F^i\,dx^j \wedge dx^k$$
$$d\Theta + (A_F) \wedge \Theta = \sum\sum_m Q^{im} dp_m \wedge dx^j \wedge dx^k - \tilde{E}\,dx^1 \wedge dx^2 \wedge dx^3.$$

Now consider $\alpha \in X = T^*M$. Since $Q_F(\alpha) \neq 0$, it follows that there exists a form $\xi = \xi_i\,dx^i \in T^*_{\pi(\alpha)}N$, so that $Q^{ij}(x_0,p_0)\xi_i\xi_j \neq 0$ where $\alpha = (x_0,p_0)$ in the canonical coordinates. One can even choose the coordinates x^i so that $\xi = dx^3$. Using the generators, Θ and $d\Theta + (A_F){\wedge}\Theta$, one can compute the reduced polar equations for the following reduced flag:

Reduced Flag Definition	**Reduced Polar Equations**
$dx^1 = dx^2 = dx^3 = 0$	?
$dx^2 = dx^3 = 0$	$dp_1 = 0$
$dx^3 = 0$	$dp_1 = dp_2 = dp_3 = 0$

Thus, the Cartan characters of this flag are $s'_0 = 0$, $s'_1 = 1$, $s'_2 = 2$, and $s'_3 = 0$. On the other hand, an admissible element at α is given by relations $dp_i = (s_{ij} + a_{ij})dx^j$ where $s_{ij} = s_{ji}$ and $a_{ij} = -a_{ji}$. The condition that Θ vanish when dp_i is replaced by $(s_{ij} + a_{ij})dx^j$ is just the condition that $a_{ij} = F^k(\alpha)$ for (i, j, k) any even permutation of $(1, 2, 3)$. The condition that $d\Theta + (A_F){\wedge}\Theta$ vanish under this substitution is that $\sum_{i,m} Q^{im}(\alpha)s_{im} = -\tilde{E}(\alpha)$.

Thus, there are 4 independent equations for the 9 unknowns $\{s_{ij}, a_{ij}\}$. It follows that the space of integral elements of $\mathcal{I}$ at α has dimension $5 = s'_1 + 2s'_2 + 3s'_3$.

By Cartan's Test, the system is therefore involutive on a neighborhood of α. Then, by the Cartan-Kähler theorem and the assumption that F is real-analytic near α it follows that there is a real-analytic integral manifold $N^3 \subset X$ of $(\mathcal{I}, L)$ passing through α. Clearly, on an open neighborhood of α, such an N^3 is the graph of a section $\omega: U \to X$ where $m = \pi(\alpha) \in U$ and $\omega_m = \alpha$. But now, by construction, $d\omega = F(\omega)$. □

Several points of this proof bear comment. The proof actually shows that $(\mathcal{I}, L)$ is involutive on the open set in X where $Q_F \neq 0$. Moreover, in the analytic category, the proof shows that the local analytic solutions of $d\omega = F(\omega)$ "depend on 2 analytic functions of 2 variables" as long as $Q_F \neq 0$ (see the Appendix concerning this terminology). Finally, straightforward computation shows that, on the open set $Q_F \neq 0$, the characteristic variety of $(\mathcal{I}, L)$, say, $\Xi \subset \mathbb{P}(L)$ is given by

$$\Xi = \{(\alpha, [\xi_i dx^i]) \mid Q^{ij}(\alpha)\xi_i\xi_j = 0\}$$

and hence is a non-trivial conic at each point.

Degenerate Cases. In order to treat the extremely degenerate case where Q_F vanishes identically, it is first helpful to know the structure of such F.

Proposition 1: *Suppose that F is such that $Q_F \equiv 0$ on X. Then F can be written in the form $F(\eta) = \alpha{\wedge}\eta + \beta$ for some $\alpha \in \Omega^1(M)$ and $\beta \in \Omega^2(M)$.*

Proof: In a canonical local coordinate system, the hypothesis that $Q_F = 0$ becomes

$$\frac{\partial F^i}{\partial p_j} = -\frac{\partial F^j}{\partial p_i}.$$

This implies that the expression $B^{ijk} = \partial^2 F^i/\partial p_j \partial p_k$ is skew-symmetric in i and j and symmetric in j and k. By the usual permutation trick, this implies $B^{ijk} \equiv 0$. Thus, F^i is affine linear in the p-variables and the jacobian matrix of $\partial F/\partial p$ is skew-symmetric. In turn, this gives

$$\begin{pmatrix} F^1 \\ F^2 \\ F^3 \end{pmatrix} = \begin{pmatrix} 0 & a^3 & -a^2 \\ -a^3 & 0 & a^1 \\ a^2 & -a^1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix}$$

for some functions $a_i(x)$, $b^i(x)$. Setting

$$\alpha = a_i dx^i$$
$$\beta = b^1\, dx^2 \wedge dx^3 + b^2\, dx^3 \wedge dx^1 + b^3\, dx^1 \wedge dx^2$$

yields $F(\eta) = \alpha{\wedge}\eta + \beta$, as desired. □

Relation with Jacobowitz's Results. This is a good place to remark on Jacobowitz' paper [*J*] on the equation $d\omega = F(\omega)$. Jacobowitz proves local solvability of this equation when F satisfies his condition ($*$) (see [*J*], pg. 362). It is easy to see that Jacobowitz' condition ($*$) in the present case of $F\colon \Lambda^1 \to \Lambda^2$ on a 3-manifold M is equivalent to the condition that F be of the form $F(\eta) = \alpha{\wedge}\eta + \beta$, where α and β satisfy $d\alpha = 0$ and $d\beta = \alpha{\wedge}\beta$.

This particular case of Jacobowitz' solvability result can be derived more simply than as in [J]. Here is how. First, apply the Poincaré Lemma to write $\alpha = da$ for some local $a \in C^\infty(U)$. Next, notice that $d(e^{-a}\beta) = 0$ so that, by the Poincaré Lemma, $\beta = e^a d\psi$ for some local 1-form ψ. The equation $d\omega = F(\omega)$ is now written in the form

$$d\omega = da\wedge\omega + e^a d\psi$$

or equivalently

$$d(e^{-a}\omega) = d\psi.$$

Thus the solutions are locally of the form $\omega = e^a(\psi + df)$ where f is an arbitrary smooth function.

On the other hand, if $d\alpha = 0$ but $d\beta \neq \alpha{\wedge}\beta$, then $d\omega = \alpha{\wedge}\omega + \beta$ has no solution, as can be seen by computing the exterior derivative of this equation.

The remaining case when $Q_F \equiv 0$ is covered by the following proposition.

Proposition 2: *If $\alpha \in \Omega^1(M)$ satisfies $d\alpha \neq 0$ and β is any 2-form on M, then the equation $d\omega = \alpha{\wedge}\omega + \beta$ is locally solvable.*

Proof: Choose local coordinates so that $d\alpha = dx^1{\wedge}dx^2$. Then $\alpha = \frac{1}{2}(x^1\,dx^2 - x^2\,dx^1) + df$ for some function f. If $d\omega = \alpha{\wedge}\omega + \beta$, then $d(e^{-f}\omega) = (\alpha - df){\wedge}(e^{-f}\omega) + e^{-f}\beta$ and conversely. Thus, it suffices to prove local solvability when $\alpha = \frac{1}{2}(x^1\,dx^2 - x^2\,dx^1)$.

Write $\omega = p_i\,dx^i$. If ω satisfies $d\omega = \alpha{\wedge}\omega + \beta$, then

$$\begin{aligned} 0 &= d\alpha\wedge\omega - \alpha\wedge\beta + d\beta \\ &= dx^1\wedge dx^2\wedge\omega + (d\beta - \alpha\wedge\beta). \end{aligned}$$

It follows that $-p_3\,dx^1{\wedge}dx^2{\wedge}dx^3 = d\beta - \alpha{\wedge}\beta$, so p_3 is determined in terms of α and β. Writing $\eta = p_1\,dx^1 + p_2\,dx^2$, the equation $d\omega = \alpha{\wedge}\omega + \beta$ is equivalent to

$$d\eta = \alpha\wedge\eta + (\beta - dp_3\wedge dx^3 + \alpha\wedge p_3\,dx^3).$$

Set $\tilde\beta = \beta - dp_3{\wedge}dx^3 + \alpha{\wedge}p_3\,dx^3$. From the relation $d\beta - \alpha{\wedge}\beta + p_3\,dx^1{\wedge}dx^2{\wedge}dx^3 = 0$, one computes that $d(\tilde\beta) = \alpha{\wedge}\tilde\beta$. Now, it is easy to see that there are functions b, q_1, and q_2, so that $\tilde\beta = b\,dx^1{\wedge}dx^2 + d(q_1\,dx^1 + q_2\,dx^2)$. Setting $\eta = \rho + (q_1\,dx^1 + q_2\,dx^2)$, the above equation is expressed as

$$d\rho = \alpha\wedge\rho + \big(\tilde\beta - d(q_1\,dx^1 + q_2\,dx^2) + \alpha\wedge(q_1\,dx^1 + q_2\,dx^2)\big)$$

or, more compactly,

$$d\rho = \alpha\wedge\rho + B\,dx^1\wedge dx^2$$

where $B = b - \frac{1}{2}(x^1\,q_1 + x^2\,q_2)$. Now, $d(b\,dx^1dx^2) = 0$ results from the equation $d\tilde\beta = \alpha{\wedge}\tilde\beta$. Thus B is a function of $\{x^1, x^2\}$. Writing $\rho = r_1\,dx^1 + r_2\,dx^2$, one sees that ρ satisfies this last equation if and only if r_1 and r_2 are functions of $\{x^1, x^2\}$ satisfying the equation

$$\frac{\partial r_2}{\partial x^1} - \frac{\partial r_1}{\partial x^2} = -\frac{1}{2}(x^1\,r_1 + x^2\,r_2) + B(x^1, x^2).$$

This last equation is obviously solvable locally (one can even specify one of r_1 or r_2 arbitrarily). Tracing such a solution ρ back through all of the changes of notation produces a local solution of $d\omega = \alpha{\wedge}\omega + \beta$. □

Return to the General Case. Note that the proof of Proposition 2 requires only ordinary differential equation theory. Upon seeing this, it is natural to wonder if the more general case where $Q_F \neq 0$ can somehow be reduced to ordinary differential equations. As an indication that this is *not* so, consider the case where $M^3 = \mathbb{R}^3$ endowed with its standard metric and orientation and where $F(\alpha) = *\alpha$. Any solution of the equation $d\omega = *\omega$ satisfies $d(*\omega) = 0$ and hence $\Delta\omega = (*d)^2\omega = \omega$. Thus $\omega = p_i\, dx^i$ where $\Delta p_i = p_i$ for $1 \leq i \leq 3$. In particular, ω is real-analytic. Ordinary differential equation methods never give such regularity unless the solutions depend only on constants.

Remarks on the Smooth Theory. To prove solvability of $d\omega = F(\omega)$ in the smooth category, it seems to be necessary to make stronger non-degeneracy assumptions than just that Q_F should not vanish. Recalling that $Q_F(\alpha)$ is naturally an element of $S^2(T_m)\otimes\Lambda^3(T^*_m)$ for $\alpha \in T^*_mM$, one says that α is F-*elliptic* if the quadratic form $Q_F(\alpha)$ is definite and that α is F-*hyperbolic* if the quadratic form $Q_F(\alpha)$ is non-degenerate but not definite. (Note that $Q_F(\alpha)$ is a quadratic form on T^*_m with values in the 1-dimensional vector space $\Lambda^3(T^*_m)$. Since the isomorphism $\Lambda^3(T^*_m) \simeq \mathbb{R}$ is not canonical, one cannot distinguish "positive" definite from "negative" definite nor can one distinguish "type (1,2)" from "type (2,1)" in the hyperbolic case.) If α is neither F-elliptic nor F-hyperbolic, α is said to be F-*degenerate*.

Here is a sketch of the proof of local solvability of $d\omega = F(\omega)$ when F has non-degenerate points: Imitating an idea of DeTurck's, replace the given pair of overdetermined equations, $d\omega = F(\omega)$ and $dF(\omega) = 0$, by an *underdetermined* system as follows: Write $\omega = \eta + df$ where η is a 1-form and f is a function. The equation $d\omega = F(\omega)$ then becomes the equation $d\eta = F(\eta + df)$. This latter equation represents a system of 3 first-order equations for the 4 unknowns consisting of f and the three coefficients of η. It is easy to see that if $\alpha \in T^*_mM$ is F-elliptic (resp. F-hyperbolic), then the linearization of the equation $d\eta - F(\eta + df) = 0$ at a pair (η_0, f_0) satisfying $(\eta_0 + df_0)|_m = \alpha$ is an underdetermined elliptic (resp. hyperbolic) system on a neighborhood of $m \in M$.

Now, it is well known that underdetermined elliptic or hyperbolic systems have solutions if they have "infinitesimal" solutions (for the elliptic case, for example, see [*D*], Theorem 2.3). However, in order to produce a solution of $d\omega = F(\omega)$ with $\omega_m = \alpha$, one must produce a solution of $d\eta = F(\eta + df)$ which satisfies the first-order condition $(\eta + df)_m = \alpha$. Since the partial differential equations to be solved are themselves of first-order, this requires some care.

The reader may recall that the usual theorems in the literature for first-order systems merely state that one can specify the value (i.e., the "0-jet") of the solution at a point and say nothing about how much of the "1-jet" of a solution one can specify at a point. For example, compare Theorem 45 in the Appendix "Sobolev Spaces and Elliptic Operators" of [Be]. However, in a private communication, Michael Taylor has shown me a proof that one *can* specify the 1-jet of a solution to a first-order elliptic or hyperbolic system arbitrarily subject only to the condition that the 1-jet actually satisfy the equation.

It follows from this (quite) general result that the equation $d\eta = F(\eta + df)$ does indeed have a solution with $(\eta + df)_m = \alpha$ whenever α is either F-elliptic or F-hyperbolic.

It is worth remarking that, in the smooth category, the local solvability of $d\omega = F(\omega)$ near an F-degenerate α for which $Q_F(\alpha) \neq 0$ remains open and may very well fail.

I close this section with the following regularity result:

Theorem 2: *If $F\colon \Lambda^1(M) \to \Lambda^2(M)$ is smooth (resp.real-analytic) and ω is a local C^1 solution of $d\omega = F(\omega)$ so that ω_m is F-elliptic for all m in the domain of ω, then ω is smooth (resp. real-analytic).*

Proof: In either case, ω is a solution of the overdetermined first-order system

$$\begin{aligned} d\omega &= F(\omega) \\ dF(\omega) &= 0. \end{aligned}$$

The linearization of this system at ω is elliptic, so the stated regularity follows (for example, see the above-mentioned Appendix in [Be]).

§B. Symmetrization of Connections on TM.

In this section, I will assume M^3 to be oriented. (Since the results are usually local, the reader may suppose M^3 to be $\mathbb{R}^3$.) The problem to be considered in this section is the following one: Given a connection ∇ on TM, when does there exist a bundle automorphism $L\colon TM \to TM$ so that the pull-back connection ∇^L is symmetric? Recall that for any connection ∇ on TM, the torsion of ∇ is the map $T_\nabla\colon \Lambda^2(TM) \to TM$ which, on vector fields X and Y on M, is defined by

$$T_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]\,.$$

The connection ∇ is said to be *symmetric* if $T_\nabla = 0$. It is easy to verify that the condition $T_{\nabla^L} = 0$ is a set of first-order equations for L. In fact, as will be seen, this is 9 equations for 9 unknowns.

To simplify the ensuing computations, it seems best to reformulate the given problem slightly. Let $M \times \mathbb{R}^3$ denote the trivial bundle over M. Since M is orientable, it is well-known that TM is trivial. A trivialization $\eta\colon TM \to M \times \mathbb{R}^3$ may simply be regarded as an $\mathbb{R}^3$-valued 1-form with all three components independent, i.e.,

$$\eta = \begin{pmatrix} \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix}$$

where $\eta^1{\wedge}\eta^2{\wedge}\eta^3 \neq 0$. There is no canonical trivialization, but given any particular trivialization $\omega\colon TM \to M \times \mathbb{R}^3$, any other can be got in the form $\eta = h\omega$ where $h\colon M \to \mathrm{GL}(3)$ is a smooth map.

A connection on $M \times \mathbb{R}^3$ is simply given by a 3-by-3 matrix of 1-forms α on M. According to the structure equations of É. Cartan, $\eta\colon TM \to M \times \mathbb{R}^3$ pulls α back to be a symmetric connection on TM if and only if η satisfies

$$d\eta + \alpha \wedge \eta = 0\,. \tag{1}$$

Obviously, the original problem is equivalent to the problem of determining when (1) has solutions η for a given α. The advantage of working with connections on the canonical trivial bundle $M \times \mathbb{R}^3$ is mainly computational: Many computations can be done via matrix multiplications and exterior differentiation rather than in the relatively clumsy covariant derivative notation. It is possible to retain an intrinsic approach and still get the computational advantages of exterior forms by passing to the frame bundle of M, but the extra abstraction then required is not easily justified.

The gauge group of $M \times \mathbb{R}^3$ is now identified with the space of smooth maps $g\colon M \to \mathrm{GL}(3)$. Such a g acts on a connection α by the usual formula $\alpha^g = g^{-1}dg + g^{-1}\alpha\, g$ and g acts on the trivialization η by the rule $\eta^g = g^{-1}\eta$. Of course, equation (1) is invariant under gauge transformations since

$$d\eta^g + \alpha^g \wedge \eta^g = g^{-1}(d\eta + \alpha \wedge \eta)\,.$$

Thus, the "connection symmetrizing" problem has a built-in invariance under the gauge group. A simple count shows that (1) is 9 first order equations for the 9 unknown coefficients of the components of η. Nevertheless, (1) cannot be put in Cauchy-Kowalewski form. It is easy to see why: Differentiating (1) yields

$$(d\alpha + \alpha \wedge \alpha) \wedge \eta = 0\,. \tag{2}$$

Let $\Phi = d\alpha + \alpha{\wedge}\alpha$ denote the curvature of α. Then (2) is a set of algebraic equations for η which are non-trivial if $\Phi \neq 0$. No system in Cauchy-Kowalewski form will yield algebraic equations for its unknowns by differentiation operations.

If $\Phi \equiv 0$, then the equation for η is obviously locally solvable, so I will henceforth consider only the more interesting cases where $\Phi \neq 0$.

Equation (2) shows how to construct examples α for which (1) has no local solutions. For example, on $M = \mathbb{R}^3$, let α be defined by

$$\alpha = \begin{pmatrix} x^2\,dx^3 & 0 & 0 \\ x^3\,dx^1 & 0 & 0 \\ x^1\,dx^2 & 0 & 0 \end{pmatrix}.$$

Then

$$\Phi = d\alpha + \alpha\wedge\alpha = \begin{pmatrix} dx^2{\wedge}dx^3 & 0 & 0 \\ (1-x^2x^3)dx^3{\wedge}dx^1 & 0 & 0 \\ dx^1{\wedge}dx^2 + x^1x^2\,dx^2{\wedge}dx^3 & 0 & 0 \end{pmatrix}.$$

Clearly, any η with $\Phi{\wedge}\eta = 0$ must have $\eta^1 \equiv 0$. Thus, in order to insure solvability of (1), one must assume something about Φ.

Definition 1: A connection α is said to have *non-degenerate curvature* at $m \in M$ if there exist $\xi \in T^*M$ and a basis $e_1, e_2, e_3 \in T_mM$ so that the matrix $(\Phi_m{\wedge}\xi)(e_1, e_2, e_3)$ is invertible.

To see what this means more explicitly, let ω be a fixed trivialization of TM and write

$$\Phi = F^1\,\omega^2\wedge\omega^3 + F^2\,\omega^3\wedge\omega^1 + F^3\,\omega^1\wedge\omega^2$$

where now, $F^i\colon M \to \{\text{3-by-3 matrices}\}$ is smooth. Then α has non-degenerate curvature at m iff there exists $\xi = \xi_i\,\omega^i_m \in T^*_m$ so that

$$\det\bigl(\xi_iF^i(m)\bigr) \neq 0.$$

Define

$$\Xi = \bigl\{(m,[\xi]) \mid \det\bigl(\xi_iF^i(m)\bigr) = 0\bigr\} \subset \mathbb{P}\bigl(T^*M\bigr).$$

It will be seen that this is the characteristic variety of the system of PDE to be solved, at least in the non-degenerate case.

(Similarly, one could define the complex characteristic variety $\Xi^{\mathbb{C}} \subset \mathbb{P}\bigl(T^*M\otimes\mathbb{C}\bigr)$ by letting the ξ_i be complex in the above definition.) Thus, α has non- degenerate curvature at m if Ξ_m is a *proper* subvariety of $\mathbb{P}\bigl(T^*M\bigr)$. It is easy to see that non-degeneracy of Φ holds on an open set in M and that, if Φ is non-degenerate at m, then $\Phi_m{\wedge}\eta_m = 0$ does not imply any linear relation among the components of η_m. Thus, counter-examples to solvability in the non-degenerate case will be harder to construct. In fact, Theorem 1 asserts that in the analytic category, non-degeneracy is a sufficient condition for solvability.

Theorem 1: *Suppose that α has non-degenerate curvature at $m_0 \in M$ and is real analytic on a neighborhood of m_0. Then there exists a real-analytic local trivialization η of TM on a neighborhood of m_0 so that $d\eta + \alpha{\wedge}\eta = 0$.*

Proof: Fix an analytic local trivialization ω on a neighborhood U of m_0 so that $\det(F^3) \neq 0$ on U. This can be done since α has non-degenerate curvature at m_0. Define

$$X = \{(m,h) \in U \times \mathrm{GL}(3) \mid \Phi_m \wedge h\omega_m = 0\}.$$

Writing $h = (h_1, h_2, h_3)$ where the h_i are columns of h, one sees that X is the codimension 3 subspace of $U \times \mathrm{GL}(3)$ defined by the equations

$$F^i(m)h_i = 0\,.$$

Note that one can solve for h_3 in terms of h_1 and h_2 on X and that the six components of h_1 and h_2 are functionally independent on X (in fact, they parametrize the fiber of $X \to U$).

Set

$$\Theta = d(h\omega) + \alpha\wedge(h\omega)$$

and compute $d\Theta = d\alpha{\wedge}(h\omega) - \alpha{\wedge}d(h\omega) = \Phi{\wedge}(h\omega) - \alpha{\wedge}\Theta$. Since $\Phi{\wedge}(h\omega) \equiv 0$ on X, this simplifies to

$$d\Theta = -\alpha\wedge\Theta.$$

It follows that the exterior system $\mathcal{I}$ on X generated by the three 2-form components of Θ is closed under exterior differentiation. As independence condition on X, take the bundle L spanned by $\{\omega^1, \omega^2, \omega^3\}$. Writing

$$\Theta = dh \wedge \omega + \big(h d\omega + \alpha \wedge (h\omega)\big)$$

it is visible that $(\mathcal{I}, L)$ is in linear form. The reduced polar equations for the reduced flag below are

Reduced Flag Definition	**Reduced Polar Equations**
$\omega^1 = \omega^2 = \omega^3 = 0$	?
$\omega^2 = \omega^3 = 0$	components of $\{dh_1\}$
$\omega^3 = 0$	components of $\{dh_1, dh_2\}$

Since the relation $F^i h_i = 0$ differentiates to show that

$$F^i dh_i \equiv 0 \mod \omega^1, \omega^2, \omega^3$$

it follows that the reduced characters are $s_0' = 0$, $s_1' = 3$, $s_2' = 3$, and $s_3' = 0$. On the other hand, an admissible element is described by relations $dh_i = h_{ij}\omega^j$ for $i = 1$ or 2 where the six $\{\, h_{ij} \mid i = 1, 2 \text{ and } j = 1, 2, 3 \,\}$ each have 3 components (since they are column vectors). The condition that these relations annihilate Θ cannot be more than 9 conditions on the quantities h_{ij} since one need only equate to zero the coefficients of three 2-forms on M. Since $9 \leq 3 \cdot 0 + 2 \cdot 3 + 1 \cdot 3$, the Modified Cartan Test (see the Appendix) shows that the system is involutive.

Since everything has been assumed to be real-analytic, the Cartan-Kähler theorem applies. Obviously an integral manifold $N^3 \subset X$ of $(\mathcal{I}, L)$ passing through some (m_0, h_0) is locally the graph of a function $h\colon U' \to \mathrm{GL}(3)$ on some neighborhood $U' \subset U$ of m_0. Moreover, this function h has the property that $\eta = h\omega$ satisfies $d\eta + \alpha \wedge \eta = 0$. $\square$

Definition 2: A connection α is said to have *hyperbolic curvature* at $m \in M$ if the equation $\det(F^i(m)\xi_i) = 0$ defines a smooth irreducible cubic curve in PT^*_m with two real circuits.

The reader may want to consult the discussion of characteristic cubics in [BGY]. The basic fact is that smooth real plane cubics are of two types: Those with one circuit, such as $y^2 = x^3 + x$, and those with two circuits, such as $y^2 = x^3 - x$. In the present situation, α has hyperbolic curvature if there exists a coframing ω near m so that for every $(\xi_1, \xi_2) \neq (0, 0)$ the polynomial

$$p(t) = \det(tF^3 + \xi_1 F^1 + \xi_2 F^2)$$

has 3 distinct real zeroes. (Compare the hyperbolicity definition in [F].) Hyperbolicity is an open condition, so if α has hyperbolic curvature at m_0 then there is a neighborhood of m_0, say U so that α has hyperbolic curvature at all the points of U.

Here is an example, Fix coordinates $\{x^i\}$ on $\mathbb{R}^3$, let λ be a constant, and define $\alpha_i = \frac{1}{2}(x^j\, dx^k - x^k\, dx^j)$ for (i, j, k) an even permutation of (1,2,3). Set

$$\alpha = \begin{pmatrix} \lambda\alpha_1 & \alpha_3 & \alpha_2 \\ \alpha_3 & \lambda\alpha_2 & \alpha_1 \\ \alpha_2 & \alpha_1 & \lambda\alpha_3 \end{pmatrix},$$

then

$$\Phi_0 = \begin{pmatrix} \lambda dx^2 \wedge dx^3 & dx^1 \wedge dx^2 & dx^3 \wedge dx^1 \\ dx^1 \wedge dx^2 & \lambda dx^3 \wedge dx^1 & dx^2 \wedge dx^3 \\ dx^3 \wedge dx^1 & dx^2 \wedge dx^3 & \lambda dx^1 \wedge dx^2 \end{pmatrix}.$$

Setting $\xi = \big(\xi_i dx^i\big)_0$, one computes

$$\det\big(\xi_i F^i(0)\big) = (2 + \lambda^3)\xi_1\xi_2\xi_3 - \lambda(\xi_1^3 + \xi_2^3 + \xi_3^3).$$

This is a smooth irreducible cubic with two circuits iff λ falls into one of the ranges $\lambda < -2$, $0 < \lambda < 1$, or $\lambda > 1$.

I now turn to a class of connections which are degenerate in the above sense, but nevertheless are of great intrinsic interest. Let $\mathbb{E}^3$ denote $\mathbb{R}^3$ endowed with its standard inner product $x \cdot y = {}^t x y$.

Definition 3: A connection α on $M \times \mathbb{E}^3$ is said to be *Euclidean* if the natural Euclidean structure on $M \times \mathbb{E}^3$ is α-parallel.

A Euclidean connection α satisfies $\alpha + {}^t\alpha = 0$ and hence the curvature Φ satisfies $\Phi + {}^t\Phi = 0$. Since the determinant of a skew-symmetric 3-by-3 matrix is zero, it follows that a Euclidean connection has degenerate curvature in the sense of Definition 1.

If η is a symmetrizing trivialization for a Euclidean connection α, then η pulls the standard metric on $M \times \mathbb{R}^3$ back to be $ds^2 = {}^t\eta \circ \eta$ and pulls α back to be the Levi-Civita connection of ds^2 on M. In terms of the original formulation, given a connection ∇ on TM (which possesses a ∇-parallel metric), one attempts to find an automorphism L which makes ∇^L symmetric. Thus, ∇^L is gauge- transformed into the Levi-Civita connection of the L-pull-back metric.

Definition 4: A Euclidean connection α has *Euclidean non- degenerate curvature* at $m \in M$ if the map $\Phi_m \colon \Lambda^2(T_m M) \to \mathfrak{so}(3)$ (= the space of 3-by-3 skew-symmetric matrices) is an isomorphism.

Equivalently, α has Euclidean non-degenerate curvature at m iff the three skew-symmetric matrices $\{F^i(m)\}$ are linearly independent.

From now on, assume that α is a Euclidean connection with Euclidean non-degenerate curvature Φ (at all points of M). It is then an easy exercise in linear algebra to show that there exists a unique trivialization $\omega \colon TM \to M \times \mathbb{E}^3$ which is orientation preserving and which satisfies

$$\Phi = \epsilon\, \omega \wedge {}^t\omega = \epsilon \begin{pmatrix} 0 & \omega^1 \wedge \omega^2 & -\omega^3 \wedge \omega^1 \\ -\omega^1 \wedge \omega^2 & 0 & \omega^2 \wedge \omega^3 \\ \omega^3 \wedge \omega^1 & -\omega^2 \wedge \omega^3 & 0 \end{pmatrix}$$

where $\epsilon = \pm 1$. (The ϵ cannot be got rid of by switching orientation.)

Now the set of Euclidean gauge transformations of $M \times \mathbb{E}^3$ is identified with the set of smooth maps $g \colon M \to \mathrm{O}(3)$. The previous formulae for α^g and Φ^g remain valid but now there is also the formula

$$\omega^g = \pm\, {}^t g\, \omega = \pm\, g^{-1}\, \omega$$

(" + " if $g \colon M \to \mathrm{SO}(3)$, " − " otherwise). It follows that the quantity

$$d\sigma^2 = {}^t\omega \circ \omega = \left(\omega^1\right)^2 + \left(\omega^2\right)^2 + \left(\omega^3\right)^2$$

is a metric on M which depends only on the gauge equivalence class of α. This metric has its own Levi-Civita connection, represented in the $d\sigma^2$-orthonormal trivialization ω by a 1-form β with values in $\mathfrak{so}(3)$ which satisfies the "symmetry" condition

$$d\omega + \beta \wedge \omega = 0.$$

Moreover, $\beta^g = g^{-1} dg + g^{-1} \beta g$ for any gauge- transformation $g \colon M \to \mathrm{O}(3)$. It follows that

$$(\alpha - \beta)^g = g^{-1}(\alpha - \beta) g.$$

Set

$$(\alpha - \beta) \wedge \omega = G \begin{pmatrix} \omega^2 \wedge \omega^3 \\ \omega^3 \wedge \omega^1 \\ \omega^1 \wedge \omega^2 \end{pmatrix}$$

where G is a 3-by-3 matrix of functions constructed from α and its first two derivatives. Substituting this formulae into the Bianchi identity $d\Phi = \Phi \wedge \alpha - \alpha \wedge \Phi$ yields

$$\omega \wedge {}^t\omega \wedge (\alpha - \beta) - (\alpha - \beta) \wedge \omega \wedge {}^t\omega = 0$$

or

$$\big({}^tG - G\big)\omega^1 \wedge \omega^2 \wedge \omega^3 = 0.$$

Thus, G is symmetric. Since

$$G^g = \pm{}^tg\ G\ g,$$

(" $+$ " if $\det(g) = +1$, " $-$ " otherwise) it follows that the tensor $\mathcal{G} = {}^t\omega \circ G \circ \omega$ is a symmetric quadratic form which (up to sign) depends only on the gauge equivalence class of α.

The Case where $\mathcal{G} \equiv 0$. Note that if $\mathcal{G} \equiv 0$, then $\alpha \equiv \beta$ and hence ω itself solves solves the symmetrization problem already. In this case,the equation $d\alpha + \alpha{\wedge}\alpha = d\beta + \beta{\wedge}\beta = \Phi = \epsilon\,\omega{\wedge}\,{}^t\omega$ shows that the metric $d\sigma^2$ has constant sectional curvature ϵ ($= \pm 1$). I will not pursue this case any further except to state the following result whose proof I leave to the reader.

For any function f on M, let f' be the 3-vector which satisfies $df = {}^tf'\omega$ and let f'' be the symmetric 3-by-3 matrix which satisfies $df' + \alpha\, f' = f''\omega$ (i.e., f'' is the Hessian matrix of f).

Proposition 1: *Let α have Euclidean non-degenerate curvature $\Phi = \epsilon\,\omega{\wedge}{}^t\omega$. Suppose further that $\mathcal{G} \equiv 0$. Then for any smooth function f, the $\mathbb{E}^3$-valued 1-form $\eta = (f\, I_3 + \epsilon f'')\omega$ satisfies $d\eta + \alpha{\wedge}\eta = 0$. Conversely, any $\mathbb{E}^3$-valued smooth 1-form η satisfying $d\eta + \alpha{\wedge}\eta = 0$ is locally of this form for some smooth function f.*

For the proof, the reader should just compute to verify the first statement and then apply the Frobenius theorem to derive its converse. In an obvious sense, the solutions of $d\eta + \alpha{\wedge}\eta = 0$ in the case where $\mathcal{G} \equiv 0$ depend locally on one function of three variables. Note that $f \equiv 1$ gives the known solution $\eta = \omega$.

The General Case. I now turn to the more interesting case where $\mathcal{G}$ is not zero.

Theorem 3: *Let α be a Euclidean connection on $M \times \mathbb{E}^3$ with Euclidean non-degenerate curvature Φ. Assume that $\mathcal{G}(m_0) \neq 0$ and that α is real-analytic on a neighborhood of $m_0 \in M$. Then there exist a neighborhood U of m_0 and a real analytic trivialization η of TM on U so that $d\eta + \alpha{\wedge}\eta = 0$.*

PROOF: By applying a (real-analytic) local Euclidean gauge transformation, one may assume that $G_{33}(m_0) \neq 0$. If there were a solution η of $d\eta + \alpha{\wedge}\eta = 0$, then one could express η in terms of ω by $\eta = s\omega$ where s is some smooth map $s\colon M \to \mathrm{GL}(3)$. The derivative of $d\eta + \alpha{\wedge}\eta = 0$ becomes $(d\alpha + \alpha{\wedge}\alpha){\wedge}\eta = 0$, or

$$\epsilon\,\omega \wedge {}^t\omega \wedge (s\omega) = 0.$$

This implies $s = {}^ts$. Moreover, one may also compute

$$\begin{aligned} 0 &= {}^t\omega \wedge \big[d(s\omega) + \alpha \wedge s\omega\big] \\ &= {}^t\omega \wedge \big[ds \wedge \omega - s\beta \wedge \omega + \alpha \wedge s\omega\big] \\ &= {}^t\omega \wedge \big[(ds + \alpha\, s - s\alpha) \wedge \omega + s(\alpha - \beta) \wedge \omega\big] \end{aligned}$$

Now, since $(ds + \alpha\, s - s\,\alpha) = {}^t(ds + \alpha\, s - s\alpha)$, the first term vanishes and the second term becomes

$$0 = {}^t\omega\, s(\alpha - \beta) \wedge \omega = \mathrm{tr}(sG)\,\omega^1 \wedge \omega^2 \wedge \omega^3$$

Thus s must also satisfy $\mathrm{tr}(sG) = 0$. With this in mind, define the manifold

$$X = \big\{\,(m, s) \in M \times \mathcal{S}_3 \ \big|\ \det(s) \neq 0,\ s = {}^ts,\ \mathrm{tr}\big(s\,G(m)\big) \neq 0\big\}\,.$$

Then X is a manifold which submerses onto M and has fibers of dimension 5 which are coördinatized by the second projection $s\colon X \to \mathcal{S}_3$ (= the space of symmetric 3-by-3 matrices). The relation

$$\mathrm{tr}\big(sG(m)\big) = G_{ij}(m)s_{ij} = 0$$

which holds on X can be solved for s_{33} in a neighborhood of the fiber over m_0 (since, by hypothesis, $G_{33}(m_0) \neq 0$).

Consider now the $\mathbb{E}^3$-valued 2-form on X

$$\begin{aligned}\Theta &= d(s\omega) + \alpha \wedge (s\omega) \\ &= ds \wedge \omega + \text{terms quadratic in } \omega\end{aligned}$$

Because of the symmetry of s,

$$d\Theta = -\alpha \wedge \Theta$$

so the ideal $\mathcal{I}$ generated by the three 2-form components of Θ, say $\{\Theta^1, \Theta^2, \Theta^3\}$ is differentially closed. These forms satisfy the relation

$$\omega^1 \wedge \Theta^1 + \omega^2 \wedge \Theta^2 + \omega^3 \wedge \Theta^3 = 0$$

since the relation $\operatorname{tr}(sG) \equiv 0$ on X implies that ${}^t\omega{\wedge}\Theta \equiv 0$ (getting this identity was the point of the above calculation). As independence condition, take L to be spanned by $\{\omega^1, \omega^2, \omega^3\}$. The system $(\mathcal{I}, L)$ is obviously in linear form. The reduced polar equations of the reduced flag below are tabulated below. (Recall that ds_{33} is a linear combination of the other ds_{ij} modulo $\omega^1, \omega^2, \omega^3$ due to the relation $\operatorname{tr}(sG) \equiv 0$.)

Reduced Flag Definition	**Reduced Polar Equations**
$\omega^1 = \omega^2 = \omega^3 = 0$	?
$\omega^2 = \omega^3 = 0$	$ds_{11} = ds_{12} = ds_{13} = 0$
$\omega^3 = 0$	$ds_{11} = ds_{12} = ds_{13} = ds_{22} = ds_{23} = 0$

(Recall, $ds_{ij} = ds_{ji}$.) It follows that the reduced characters are $s'_0 = 0$, $s'_1 = 3$, $s'_2 = 2$, and $s'_3 = 0$.

Now an integral element of $(\mathcal{I}, L)$ is defined by a relation of the form $ds_{ij} = s_{ijk}\omega^k$ where the $s_{ijk} = s_{jik}$ are subject to the relations coming from $d\big(\operatorname{tr}(Gs)\big) = 0$. This reduces the number of independent s_{ijk}'s to 15. These 15 are subject only to the condition of annihilating the three 2-forms Θ^i while these 2-forms are subject to the relation $\omega^i{\wedge}\Theta^i \equiv 0$. It follows that being an integral element of $\mathcal{I}$ imposes at most $N = 8$ conditions on the s_{ijk}'s. Since $8 \leq 2 \cdot 3 + 1 \cdot 2$, the Modified Cartan Test implies that $(\mathcal{I}, L)$ is involutive, with characters $s_0 = 0$, $s_1 = 3$, $s_2 = 2$, and $s_3 = 0$. Theorem 3 now follows from the Cartan-Kähler theorem applied to $(\mathcal{I}, L)$. $\square$

The characteristic variety of the system $\mathcal{I}$ is given by

$$\Xi = \big\{\, (m, [\xi]) \mid \xi = \xi_i(\omega^i)_m,\ G_{ij}(m)\xi_i\xi_j = 0 \big\} \subset \mathbb{P}\big(T^*M\big).$$

In particular, $\Xi_m \subset \mathbb{P}\big(T^*_m M\big)$ is a conic whenever $\mathcal{G}(m) \neq 0$, in accordance with Fact 1 of the Appendix. It follows from Fact 2 of the Appendix that whenever $\mathcal{G}(m)$ is definite and α is real-analytic near m, then any solution η of $d\eta + \alpha{\wedge}\eta = 0$ is real-analytic near m also.

Assume now that $\det(G)$ is nowhere vanishing on M. Due to the above transformation law for G, one may choose $g\colon M \to \mathrm{O}(3)$, so that $\det\big(G^g(m)\big) > 0$. Hence, I may assume from now on that $\det(G) > 0$. This reduces the allowable gauge transformations to those of the form $g\colon M \to \mathrm{SO}(3)$.

I will say that α is *elliptic* at m if $\mathcal{G}(m)$ is positive definite. If α is elliptic at all points of M, one can define three vector bundles over M as follows:

- Let $E_1 \subset \operatorname{Hom}\big(TM, \mathbb{E}^3\big)$ denote the (rank 5) vector bundle of $\mathbb{E}^3$-valued 1-forms η on M which satisfy $\eta = s\omega$ where s is a 3-by-3 matrix satisfying $s = {}^ts$ and $\operatorname{tr}(sG) = 0$. (By the above calculations, any solution of $d\eta + \alpha{\wedge}\eta = 0$ must be a section of E_1.)
- Let $E_2 \subset \operatorname{Hom}\big(\Lambda^2(TM), \mathbb{E}^3\big)$ denote the (rank 8) vector bundle of $\mathbb{E}^3$-valued 2-forms Ω on M which satisfy the condition ${}^t\omega{\wedge}\Omega = 0$.

- Let $E_3 = Hom(\Lambda^3(TM), \mathbb{E}^3)$ denote the (rank 3) bundle of $\mathbb{E}^3$-valued 3-forms on M.

  Define differential operators $D_1 \colon \Gamma(E_1) \to \Gamma(E_2)$ and $D_2 \colon \Gamma(E_2) \to \Gamma(E_3)$ by

$$D_1\eta = d\eta + \alpha \wedge \eta$$
$$D_2\Omega = d\Omega + \alpha \wedge \Omega.$$

Of course, one must check that, for any $\eta \in \Gamma(E_1)$, one has $D_1\eta \in \Gamma(E_2)$. However, writing $\eta = s\omega$ and using $s = {}^t s$ and $\operatorname{tr}(sG) = 0$, the identity ${}^t\omega \wedge D_1(s\omega) = 0$ is immediate from the definitions. Similarly, one checks that $D_2 \circ D_1(s\omega) \equiv 0$ for s satisfying the above conditions. The symbols of these operators are

$$\sigma_\xi(D_1)\,\eta = \xi \wedge \eta$$
$$\sigma_\xi(D_2)\,\Omega = \xi \wedge \Omega.$$

By dimension count and the obvious surjectivity of $\sigma_\xi(D_2)$, it suffices to show that $\sigma_\xi(D_1)$ is injective for $\xi \neq 0$ in order to conclude that the $\bigl\{D_i \colon \Gamma(E_i) \to \Gamma\bigl(E_{i+1}\bigr)\bigr\}$ form an elliptic complex. However, if $\sigma_\xi(D_1)(s\omega) = \xi \wedge s\omega = 0$ then $s\omega = e\xi$ for some $e \in E^3$. It follows that s has rank 0 or 1 and hence, if $s \neq 0$, then $s = \pm(e\,{}^t e)$ for some non-zero e. But then $\pm\operatorname{tr}(sG) = \operatorname{tr}\bigl(e\,{}^t eG\bigr) = {}^t eGe = 0$, so e is a null vector of G. This is impossible, since G was assumed positive definite. Thus, for elliptic α, it follows that the complex

$$0 \longrightarrow E_1 \xrightarrow{D_1} E_2 \xrightarrow{D_2} E_3 \longrightarrow 0$$

is elliptic. In fact, more is true. The reader familiar with Spencer's theory of overdetermined equations (see [Sp] or [Go]) will notice that the proof of Theorem 2 actually shows that the above sequence is *formally exact* at E_2 and E_3. I shall not have need of this result however, so I will not go into details. I could, however, use this sequence to prove a smooth solvability result for elliptic α.

§C. Connections with Prescribed Curvature.

As in previous sections, let M be a fixed 3-manifold. For any Lie group G with Lie algebra $\mathfrak{g}$, consider a principal G-bundle $P \to M$. Let $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$ denote the associated bundle of Lie algebras. The space of connections on P is an affine space modeled on $\Omega^1(\mathfrak{g}_P) \simeq \Omega^1(M) \otimes \mathfrak{g}_P$. For each connection A, there is the associated curvature $F(A)$, which is a section of $\Omega^2(\mathfrak{g}_P) \simeq \Omega^2(M) \otimes \mathfrak{g}_P$. The operator $A \mapsto F(A)$ is a first-order non-linear operator which is, in the naïve sense, determined. Since the discussion in this section is local, one may as well set $P = M \times G$ and represent the connection A by a $\mathfrak{g}$-valued 1-form α. Then $F(A)$ is represented by $\Phi = d\alpha + \frac{1}{2}[\alpha, \alpha]$. The problem to be discussed in this section is this:

Given a $\Phi \in \Omega^2(\mathfrak{g})$, when does there exist an $\alpha \in \Omega^1(\mathfrak{g})$ so that

$$d\alpha + \tfrac{1}{2}[\alpha, \alpha] = \Phi? \tag{1}$$

Expressing the problem in this way, it is apparent that this is $3 \dim \mathfrak{g}$ equations for $3 \dim \mathfrak{g}$ unknowns. Of course, this system of equations is "degenerate" since its principal symbol is the same as that of the exterior derivative.

In the special case $\Phi = 0$, the local solutions of (1) are of the form $\alpha = g^{-1} dg$ where $g \colon M \to G$ is a smooth map. However, when $\Phi \neq 0$, these equations are not so readily solved (and the space of solutions is not generally so large).

Differentiating (1), yields the Bianchi identity

$$[\Phi, \alpha] = d\Phi \tag{2}$$

which represents a set of at most $\dim \mathfrak{g}$ equations of order zero for α.

Now, one may not always be able to solve these equations, even algebraically. For example, in the extreme case where $\mathfrak{g}$ is abelian, (2) implies that Φ must be closed if (1) is to be solvable. Apparently, one must impose suitable non-degeneracy hypotheses on Φ in order to achieve solvability of (1).

Even at the "opposite extreme," when $\mathfrak{g}$ is semi-simple, (2) can represent an obstruction to solvability. For example, let $G = \mathrm{SL}(3, R)$, let $M = \mathbb{R}^3$, and let

$$\Phi = \begin{pmatrix} dx^2 \wedge dx^3 & dx^1 \wedge dx^2 & 0 \\ dx^3 \wedge dx^1 & x^1 dx^2 \wedge dx^3 & 0 \\ 0 & 0 & -(1 + x^1) dx^2 \wedge dx^3 \end{pmatrix}.$$

Then, even though Φ_m is injective for all m when regarded as a map $\Lambda^2\bigl(T_m M\bigr) \to \mathfrak{sl}(3, R)$ (= trace-zero 3-by-3 matrices), there is no solution α to the equation (2) along the hypersurface $x^1 = -1$. Thus, to ensure solvability of (1), it is not enough to suppose, when Φ is written in the form $\Phi = F^1 \omega^2 \wedge \omega^3 + F^2 \omega^3 \wedge \omega^1 + F^3 \omega^1 \wedge \omega^2$, that the elements $\{F^i(m) \mid i = 1, 2, 3\}$ are linearly independent.

It seems that a reasonable hypothesis is that the spaces $\mathrm{ad}(F^i)(\mathfrak{g}) \subset \mathfrak{g}$ should fill out all of $\mathfrak{g}$, i.e.,

$$\mathfrak{g} = \mathrm{ad}(F^1)(\mathfrak{g}) + \mathrm{ad}(F^2)(\mathfrak{g}) + \mathrm{ad}(F^3)(\mathfrak{g}). \tag{$*$}$$

Then, at least (2) can always be solved. In fact, choosing local coordinates x^1, x^2, x^3 and writing

$$\Phi = F^1 \, dx^1 \wedge dx^2 + F^2 \, dx^3 \wedge dx^1 + F^3 \, dx^1 \wedge dx^2,$$

equation (2) becomes an equation for $\alpha = A_i \, dx^i$ of the form

$$[F^1, A_1] + [F^2, A_2] + [F^3, A_3] = \partial_1 F^1 + \partial_2 F^2 + \partial_3 F^3 \tag{$2'$}$$

With the hypothesis $(*)$, the space of solutions of $(2')$ forms a natural affine space of dimension $2 \dim \mathfrak{g}$ at each point.

However, in general, $(2')$ is not the only set of zeroth order equations which α must satisfy. To see this, suppose that $\mathfrak{g}$ is simple and let $\langle,\rangle$ denote the Cartan-Killing form of $\mathfrak{g}$. Writing

$$dA_i = (s_{ij} + a_{ij})dx^j$$

where $s_{ij} = s_{ji}$ and $a_{ij} = -a_{ji}$, equation (1) is expressible in the form

$$a_{ij} = \tfrac{1}{2}\big([A_i, A_j] - F^k\big)$$

where (i, j, k) is an even permutation of (1,2,3). On the other hand, differentiating (2) yields

$$\begin{aligned}[F^i, s_{ij}] &= -[F^i, a_{ij}] - \partial^2_{ij}F^i - [\partial_j F^i, A_i]\\ &= -\tfrac{1}{2}[F^i, [A_i, A_j] - F^k] - \partial^2_{ij}F^i - [\partial_j F^i, A_i].\end{aligned}$$

Using the ad-invariance of $\langle,\rangle$ and summing on i and j then yields

$$\langle F^j, [F^i, s_{ij}]\rangle = -\langle [F^i, F^j], s_{ij}\rangle = 0$$

due to the symmetries $s_{ij} = s_{ji}$ and $[F^i, F^j] = -[F^j, F^i]$. It follows that there exists a relation of the form

$$\langle [F^i, F^j], [A_i, A_j]\rangle = \text{expression in } \big\{F^i, \partial_j F^i, \partial^2_{ij}F^i, A_i\big\}. \tag{3}$$

This relation is non-trivial for $\mathfrak{g} = \mathfrak{so}(3)$. It will be shown below that for "generic" Φ with values in $\mathfrak{so}(3)$, there are no further zeroth order relations on α derivable by differentiation of (1).

There is evidence for the following conjecture: If $\mathfrak{g}$ is simple of dimension d and rank r and $\Phi \in \Omega^2(\mathfrak{g})$ is in general position at m (i.e., the three components $F^i(m)$ are in general position in $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$), then any solution $\alpha = A_i dx^i$ of (1) on a neighborhood of m satisfies not only the d linear, zeroth order relations (2) given by the Bianchi identity, but also r additional non-linear, zeroth-order relations. Moreover the first-order system (1) adjoined to the $(d+r)$ relations of order zero forms an involutive system with Cartan characters $s_1 = d$, $s_2 = d - r$, and $s_3 = 0$.

Unfortunately, these "hidden Bianchi identities" tend to be rather complicated as soon as $r > 1$. Since the theory in the higher rank case is unsatisfactory at present, I will content myself with a discussion of the simplest case: $\mathfrak{g} = \mathfrak{so}(3)$.

To my knowledge, local solvability of (1) for generic $\Phi \in \Omega^2\big(\mathfrak{so}(3)\big)$ was first shown in the analytic category by Tsarev [T]. Dennis DeTurck informs me that he has proved this local solvability result in the smooth category as well. On the one hand, Tsarev does not discuss the characteristics of the system, while DeTurck shows that the problem can be studied by elliptic methods. The purpose of the following discussion is to reprove and sharpen Tsarev's result, to discuss the characteristic variety, and to show that one can sometimes reduce the problem to either a hyperbolic one *or* an elliptic one.

In what follows, it will be useful to identify $\mathfrak{so}(3)$ with $\mathbb{E}^3$ by the usual cross-product convention. Thus, for $x = (x^i) \in \mathbb{E}^3$, let $[x]$ denote the 3-by-3 skew-symmetric matrix representing "cross-product with x." In symbols,

$$[x] = \begin{pmatrix} 0 & x^3 & -x^2 \\ -x^3 & 0 & x^1 \\ x^2 & -x^1 & 0 \end{pmatrix}.$$

Regarding α and Φ as $\mathbb{E}^3$-valued forms, equation (1) becomes

$$d\alpha + \tfrac{1}{2}[\alpha] \wedge \alpha = \Phi.$$

Let $\Phi = (\Phi^i)$ be given and assume that the 2-forms $\{\Phi^1, \Phi^2, \Phi^3\}$ are linearly independent on M. It then follows, just as in §B, that there is a coframing $\omega = (\omega^i)$ so that

$$\Phi = \epsilon \begin{pmatrix} \omega^2\wedge\omega^3 \\ \omega^3\wedge\omega^1 \\ \omega^1\wedge\omega^2 \end{pmatrix} = -\tfrac{1}{2}\epsilon\,[\omega]\wedge\omega$$

where $\epsilon = \pm 1$ and ω is unique up to replacement by $-\omega$. In this new notation, the formulae for the action of a gauge transformation $g\colon M \to \mathrm{SO}(3)$ on Φ and on ω are

$$\begin{aligned} \Phi^g &= g^{-1}\Phi \\ \omega^g &= g^{-1}\omega \end{aligned}$$

Just as before, it follows that $d\sigma^2 = {}^t\omega \circ \omega$ is a metric on M which depends only on the gauge equivalence class of Φ. Note that $d\sigma^2$ can be characterized as the unique metric on M which makes $\Phi_m\colon \Lambda^2(T_mM) \to \mathfrak{so}(3)$ into an isometry for all $m \in M$ (assuming that a suitable multiple of the Cartan-Killing form on $\mathfrak{so}(3)$ has been fixed). In fact, it is easy to see that two non-degenerate $\mathfrak{so}(3)$-valued 2-forms Φ_1 and Φ_2 are gauge equivalent on M if and only if their invariants satisfy $\epsilon_1 = \epsilon_2$ and $d\sigma_1^2 = d\sigma_2^2$.

The Levi-Civita connection form β (with values in $\mathbb{E}^3$) associated to the orthonormal coframe ω is defined by the condition

$$d\omega = -[\beta] \wedge \omega.$$

Defining the matrix $R = (R^{ij})$ by the equation

$$d\beta + \tfrac{1}{2}\,[\beta] \wedge \beta = -\tfrac{1}{2}\,R\,[\omega] \wedge \omega,$$

then the first Bianchi identity for the metric $d\sigma^2$ takes the form $R = {}^tR$. Unwinding the definitions, one gets

$$\mathrm{Riem}\left(d\sigma^2\right) = R^{i_1 j_2}\left(\omega^{i_2} \wedge \omega^{i_3}\right) \circ \left(\omega^{j_2} \wedge \omega^{j_3}\right)$$

where, in this formula, (i_1, i_2, i_3) and (j_1, j_2, j_3) are to be summed over even permutations of (1,2,3). Thus, the scalar curvature r of $d\sigma^2$ is given by the formula $r = 2\big(R^{11} + R^{22} + R^{33}\big)$.

Now suppose that α were a solution of the equation $d\alpha + \frac{1}{2}[\alpha]\wedge\alpha = \Phi = -(\epsilon/2)[\omega]\wedge\omega$. The Bianchi identity (2) then reduces to

$$[\alpha - \beta] \wedge [\omega] \wedge \omega = 0.$$

This, in turn, is equivalent to $\alpha - \beta = S\omega$ where S is a symmetric 3-by-3 matrix. Computing the exterior derivative of this relation yields

$$\begin{aligned} 0 &= d\alpha - d\beta - d(S\omega) \\ &= -\tfrac{1}{2}[\alpha] \wedge \alpha - \tfrac{1}{2}\epsilon[\omega] \wedge \omega + \tfrac{1}{2}[\beta] \wedge \beta + \tfrac{1}{2}R[\omega] \wedge \omega - dS \wedge \omega + S\,[\beta] \wedge \omega \\ &= -\tfrac{1}{2}[\beta + S\omega] \wedge (\beta + S\omega) + \tfrac{1}{2}[\beta] \wedge \beta + \tfrac{1}{2}(R - \epsilon\, I_3)[\omega] \wedge \omega - dS \wedge \omega + S[\beta] \wedge \omega. \end{aligned}$$

Thus

$$0 = -\big(dS + [\beta]S - S[\beta]\big) \wedge \omega - \tfrac{1}{2}[S\omega] \wedge S\omega + \tfrac{1}{2}(R - \epsilon \wedge I_3)[\omega] \wedge \omega.$$

Wedging this relation with ${}^t\omega$ in front, the first term vanishes due to the symmetry of S and the remaining terms are easily computed:

$$\begin{aligned} {}^t\omega \wedge [S\omega] \wedge S\omega &= \big(\mathrm{tr}(S^2) - (\mathrm{tr}\, S)^2\big)\, \omega^1 \wedge \omega^2 \wedge \omega^3 \\ {}^t\omega \wedge \big(R - \epsilon I_3\big)[\omega] \wedge \omega &= 6\epsilon - r. \end{aligned}$$

Thus, for any solution α, the following relation holds:

$$\big(\mathrm{tr}\, S\big)^2 - \mathrm{tr}(S^2) = r - 6\epsilon. \tag{4}$$

Equation (4) is the $\mathfrak{so}(3)$-manifestation of (3). This relation must hold in addition to the three Bianchi relations: $S = {}^tS$ (i.e., the $\mathfrak{so}(3)$-manifestation of (2)).

At this point a more precise result than Tsarev's can be stated.

Theorem : *Suppose that Φ is such that $d\sigma^2$ is real-analytic in some coordinate system about $m_0 \in M$. Then one may choose a gauge transformation which makes Φ real-analytic near m_0. Write $\Phi = -(\epsilon/2)[\omega]{\scriptstyle\wedge}\omega$ and define β and r as above. Then for any non-zero 3-by-3 symmetric matrix S_0 satisfying $\bigl(\mathrm{tr}\, S_0\bigr)^2 - \mathrm{tr}(S_0^2) = r(m_0) - 6\epsilon$, there exists a connection α on a neighborhood of m_0 so that $\alpha_{m_0} = (\beta + S_0\omega)_{m_0}$ and $d\alpha + \frac{1}{2}[\alpha]{\scriptstyle\wedge}\alpha = \Phi$ on that neighborhood. Moreover, if the matrix $G_0 = -(\mathrm{tr} S_0)I_3 + S_0$ is either positive or negative definite, then any α satisfying the above two conditions is real- analytic on a neighborhood of m_0.*

Proof: If $d\sigma^2$ is real-analytic near m_0, one can clearly choose a $d\sigma^2$-orthonormal, real-analytic coframing ω near m_0. Now throw away the part of M outside this neighborhood. Applying a gauge transformation to Φ one may assume that $\Phi = -(\epsilon/2)[\omega]{\scriptstyle\wedge}\omega$. Now let X be the manifold of pairs (m, S) where $m \in M$ and S is a *non-zero* symmetric 3-by- 3 matrix satisfying $(\mathrm{tr}\, S)^2 - \mathrm{tr}(S^2) = r(m) - 6\epsilon$. That X is a smooth manifold follows easily from the fact that the quadratic form $Q(S) = (\mathrm{tr} S)^2 - \mathrm{tr}(S^2)$ is non-degenerate on the space of symmetric matrices. On X, define the $\mathbb{E}^3$-valued 2-form

$$\Theta = d(S\omega + \beta) - \tfrac{1}{2}[S\omega + \beta] \wedge (S\omega + \beta) + \tfrac{1}{2}\epsilon\, [\omega] \wedge \omega.$$

By construction, ${}^t\omega{\scriptstyle\wedge}\Theta \equiv 0$ and the ideal $\mathcal{I}$ generated by the 2-forms $\{\Theta^1, \Theta^2, \Theta^3\}$ is differentially closed. The terms dS^{ij} are not linearly independent modulo $\{\omega^i\}$, but satisfy, in addition to the relations $dS^{ij} = dS^{ji}$, the non-trivial relation

$$\mathrm{tr}\Bigl(\bigl((\mathrm{tr} S)I_3 - S\bigr)dS\Bigr) - \tfrac{1}{2}\, dr \equiv 0 \mod \omega^1, \omega^2, \omega^3. \tag{5}$$

The reduced characters are easily computed to be

$$s_0' = 0, \quad s_1' = 3, \quad s_2' = 2, \quad s_3' = 0.$$

On the other hand, the condition ${}^t\omega{\scriptstyle\wedge}\Theta = 0$ shows that the space of integral elements of $(\mathcal{I}, L)$ has codimension at most 8 in the space of admissible elements. Since $8 \le 3\cdot 0 + 2\cdot 3 + 1\cdot 2 + 0\cdot 0$, it follows by Cartan's Test that $(\mathcal{I}, L)$ is involutive. By the Cartan-Kähler Theorem, integral manifolds of $(\mathcal{I}, L)$ exist passing through (m_0, S_0).

Finally, the relation (5) shows that (I, L) is elliptic in the region where $(\mathrm{tr} S)I_3 - S$ is positive or negative definite. $\square$

Set $G = S - (\mathrm{tr}\ S)I_3$. Then G has the same significance for α as it did in §B (in fact, it is the same quantity). The quadratic relation (4) then becomes

$$\tfrac{1}{2}(\mathrm{tr}\, G)^2 - \mathrm{tr}\,(G^2) = r - 6\epsilon. \tag{4'}$$

It follows that for any value of $r(m)$ there exists a positive definite solution G of (4′). Thus, for any analytic Φ one can always locally solve (1) for an elliptic α. Moreover, if $r(m) > 6\epsilon$, then *every* solution G of (4′) is either positive or negative definite. The reason is simple: If G has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ then (4′) is just $p(\lambda) = r - 6\epsilon$ where

$$p(\lambda) = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3)^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2).$$

However, $p(\lambda) > 0$ implies that all the λ_i have the same sign.

As a corollary of the above calculations, it follows that if $d\sigma^2$ satisfies $r > 6\epsilon$ and is analytic, then up to gauge transformations, all solutions α of (1) are also real-analytic.

On the other hand, if $r(m) < 6\epsilon$ there exist solutions G of all ranks and signatures. It is not difficult to see that, if $r < 6\epsilon$, this sort of analyticity result need not hold.

§D. Prescribed Riemann Curvature.

In this section, I want to consider the solvability of a second-order operator on M^3, the Riemann curvature operator. In general, for a metric ds^2 on an n-manifold M^n, the Riemann curvature tensor of ds^2 *with all of its indices lowered* takes values in the subbundle $\mathcal{K} \subset S^2(\Lambda^2 T^*M)$ which is the kernel of the natural wedge-product map $S^2(\Lambda^2 T^*M) \to \Lambda^4 T^*M$ (this is merely the first Bianchi identity). Thus, there is a natural second- order non-linear differential operator $\mathrm{Riem}\colon \Gamma\big(S^2_+(T^*)\big) \to \Gamma\big(\mathcal{K}\big)$. In general, the rank of $\mathcal{K}$ is much larger than the rank of $S^2(T^*)$, but for $n = 3$, these bundles are both of rank 6 over M, so the equation $\mathrm{Riem}(ds^2) = \mathcal{R}$ for $\mathcal{R} \in \Gamma\big(\mathcal{K}\big)$ is "determined" in the naive sense. (When $n = 3$, note that $\mathcal{K} = S^2(\Lambda^2 T^*M)$ since $\Lambda^4 T^*M = (0)$.)

To fix notation, I will briefly review the structure equations for a metric ds^2. At first, n can be arbitrary. Let $\omega = (\omega^i)$ be a local coframing on an open set $U \subset M$ (think of ω as a column of 1-forms). Then

$$ds^2|_U = G_{ij}\omega^i \circ \omega^j = {}^t\omega\, G\, \omega$$

where $G = (G_{ij}) > 0$ is a positive definite matrix of functions on U.

It is important to understand how these formulae change when a new coframing is selected. If $g\colon U \to \mathrm{GL}(n)$ is arbitrary, a new coframing can be writ in the form $\omega^g = g^{-1}\omega$. Under this change-of-frame, the new coefficient matrix becomes $G^g = {}^t g\, G\, g$.

The Levi-Civita connection γ on U is the n-by- n matrix of 1-forms $\gamma = (\gamma^i_j)$ satisfying

$$\begin{aligned} d\omega &= -\gamma \wedge \omega \\ dG &= {}^t\gamma G + G\gamma. \end{aligned}$$

The formula $\gamma^g = g^{-1}dg + g^{-1}\gamma g$ shows how γ changes with a change of frame. The curvature of γ is $d\gamma + \gamma\wedge\gamma$. Set

$$\Omega = G(d\gamma + \gamma \wedge \gamma) = (\Omega_{ij}).$$

Then $\Omega^g = {}^t g\, \Omega\, g$ and computing $d^2\omega = d^2 G = 0$ yields

$$\begin{aligned} \Omega \wedge \omega &= 0 \\ \Omega + {}^t\Omega &= 0. \end{aligned}$$

Thus, $\Omega_{ij} = -\Omega_{ij}$, and these 2-forms can be expanded as

$$\Omega_{ij} = \tfrac{1}{2} R_{ijk\ell}\omega^k \wedge \omega^\ell$$

where R has the usual symmetries of the Riemann curvature tensor. The change-of-frame rules for ω, G, etc., show that the quantity

$$\mathrm{Riem}\big(ds^2|_U\big) = \tfrac{1}{4} R_{ijk\ell}(\omega^i \wedge \omega^j) \circ (\omega^k \wedge \omega^\ell)$$

is independent of the choice of ω so there is a well-defined map $\mathrm{Riem}\colon \Gamma\big(S^2_+(T^*M)\big) \to \Gamma(\mathcal{K})$ as described above. (The factor of $\frac{1}{4}$ was chosen so that for the standard unit 2-sphere, $\mathrm{Riem}(ds^2_0) = (vol_0)^2$, where vol_0 is the area form of ds^2_0.)

In dimension 3, Cartan's "omitted index" notation is useful. Set $K^{i_1 j_1} = R_{i_2 i_3 j_2 j_3}$ where (i_1, i_2, i_3) and (j_1, j_2, j_3) are even permutations of (1,2,3). Then consider the matrix $K = (K^{ij}) = {}^tK$. The change-of-frame rule for K is easily seen to be

$$K^g = \big(\det(g)\big)^2 g^{-1}\, K\, {}^t g^{-1}.$$

The formula for the Riemann curvature tensor can then be rewritten as

$$\mathrm{Riem}(ds^2|_U) = K^{i_1 j_1}(\omega^{i_2} \wedge \omega^{i_3}) \circ (\omega^{j_2} \wedge \omega^{j_3})$$

where the right hand side is to be summed over all (i_1, i_2, i_3) and (j_1, j_2, j_3) which are cyclic permutations of (1,2,3). Moreover, setting $\Omega^{i_1} = \Omega_{i_2 i_3}$ yields

$$\Omega^i = K^{ij_1}\omega^{j_2}\wedge\omega^{j_3}.$$

The second Bianchi identity, $d\Omega = {}^t\gamma\wedge\Omega + \Omega\wedge\gamma$, can now be written easily in terms of K as

$$\big(dK + \gamma\wedge K + K\wedge{}^t\gamma - 2\,\mathrm{tr}(\gamma)K\big)\wedge\begin{pmatrix}\omega^2\wedge\omega^3\\ \omega^3\wedge\omega^1\\ \omega^1\wedge\omega^2\end{pmatrix}.$$

Now suppose that a section $\mathcal{K}\in\Gamma(\mathscr{K}) = \Gamma\big(S^2(\Lambda^2T^*M)\big)$ and been specified. The problem to be discussed to determine whether there is a metric ds^2 so that $\mathrm{Riem}(ds^2) = \mathcal{K}$.

It is easy to see that, without some non-degeneracy hypothesis, this equation need not be solvable. For example, let $\mathcal{K} = x^1(dx^2\wedge dx^3)^2 + x^2(dx^3\wedge dx^1)^2 + x^3(dx^1\wedge dx^2)^2$ on a neighborhood of the origin in $\mathbb{R}^3$ and let $\omega^1 = dx^i$. Then no metric $ds^2 = g_{ij}dx^i\circ dx^j$ can satisfy $\mathrm{Riem}(ds^2) = \mathcal{K}$ on a neighborhood of the origin since, no matter what γ is, it cannot satisfy the above second Bianchi identity. (This example is essentially due to Dennis DeTurck.)

The remainder of the discussion in this section will concern the case where $\mathcal{K}$ is assumed to be a positive definite section of $\Gamma\big(S^2(\Lambda^2T^*M)\big)$. (The other non-degenerate cases behave exactly in the same way except that one must keep track of a few signs when one raises and lowers indices. I leave this to the reader.) Under these assumptions, one can always choose a local coframing ω so that

$$\mathcal{K} = (\omega^2\wedge\omega^3)^2 + (\omega^3\wedge\omega^1)^2 + (\omega^1\wedge\omega^2)^2.$$

(Remember, the squares are symmetric, not exterior.) This ω is unique up to a change of frame of the form $\omega^g = g^{-1}\omega$ where $g\colon U\to \mathrm{O}(3)$. If $ds^2 = {}^t\omega G\omega$ were a solution to the equation $\mathrm{Riem}\big(ds^2\big) = \mathcal{K}$ on U, then the following structure equations would hold

$$\begin{aligned} dG &= {}^t\gamma G + G\gamma\\ d\omega &= -\gamma\wedge\omega\\ G(d\gamma + \gamma\wedge\gamma) &= \Omega = \omega\wedge{}^t\omega, \end{aligned}$$

since, in this coframing, $K^{ij} = \delta^{ij}$, so $\Omega^{i_1} = \omega^{i_2}\wedge\omega^{i_3}$, so $\Omega_{ij} = \omega^i\wedge\omega^j$, so $\Omega = \omega\wedge{}^t\omega$. Moreover, since $K\equiv I_3$, the second Bianchi identity takes the form

$$\big(\gamma + {}^t\gamma - 2\mathrm{tr}(\gamma)\,I_3\big)\wedge\begin{pmatrix}\omega^2\wedge\omega^3\\ \omega^3\wedge\omega^1\\ \omega^1\wedge\omega^2\end{pmatrix} = 0.$$

The above equations are now a system of partial differential equations for the matrix $G = (G_{ij})$. For some strange reason, it turns out to be more helpful to consider, instead of G, the matrix $H = G^{-1}$. In terms of H, the equations become

$$\begin{aligned} dH &= -H\,{}^t\gamma - \gamma\,H\\ d\omega &= -\gamma\wedge\omega\\ d\gamma &= -\gamma\wedge\gamma + H\omega\wedge{}^t\omega. \end{aligned}$$

Of course, the second Bianchi identity remains the same.

Now consider the coframing ω. It has its own Levi-Civita connection χ satisfying the conditions

$$\begin{aligned} d\omega &= -\chi\wedge\omega\\ 0 &= \chi + {}^t\chi, \end{aligned}$$

The formula for the effect of the change-of-frame $g\colon U \to \mathrm{O}(3)$ on χ is $\chi^g = g^{-1}dg + g^{-1}\,\chi\, g$. Since the structure equations give

$$(\gamma - \chi) \wedge \omega = -d\omega + d\omega = 0,$$

it follows (by Cartan's Lemma) that there exist functions $F^i_{jk} = F^i_{kj}$ so that

$$\gamma^i_j - \chi^i_j = F^i_{jk}\omega^k.$$

Since

$$dH + H\,{}^t\chi + \chi\, H = -H\,{}^t(\gamma - \chi) - (\gamma - \chi)\, H$$

it follows that $\gamma - \chi$ essentially measures the covariant derivative of H with respect to the connection χ. Thus, the F^i_{jk} may be regarded as the "derivatives" of H^{ij} after some suitable index juggling. Certainly, the 1-jet of a metric ds^2 at $m_0 \in U$ is exactly determined by the quantities $\{H^{ij}(m_0), F^i_{jk}(m_0)\}$. The second Bianchi identity can now be written in the form

$$\begin{aligned} 0 &= \big(\gamma + {}^t\gamma - 2tr(\gamma)\, I_3\big) \wedge \begin{pmatrix} \omega^2 \wedge \omega^3 \\ \omega^3 \wedge \omega^1 \\ \omega^1 \wedge \omega^2 \end{pmatrix} \\ &= \big((\gamma - \chi) + {}^t(\gamma - \chi) - 2\mathrm{tr}(\gamma - \chi)\, I_3\big) \wedge \begin{pmatrix} \omega^2 \wedge \omega^3 \\ \omega^3 \wedge \omega^1 \\ \omega^1 \wedge \omega^2 \end{pmatrix} \end{aligned}$$

which is easily seen to reduce to the three equations

$$\sum_k \big(F^k_{ki} - F^i_{kk}\big) = 0$$

Of course, these represent three first-order conditions which G must satisfy if it is to satisfy the second-order equations $\mathrm{Riem}({}^t\omega\, G\, \omega) = \mathcal{K}$.

I am now going to show that there is yet another such first-order condition. To see this, set $\varphi = \gamma - \chi$ and expand the equation $d\gamma + \gamma \wedge \gamma - H\,\omega \wedge {}^t\omega = 0$ in the form

$$(d\varphi + \chi \wedge \varphi + \varphi \wedge \chi) + \varphi \wedge \varphi + (d\chi + \chi \wedge \chi) - H\, \omega \wedge {}^t\omega = 0.$$

Now, the term $d\chi + \chi \wedge \chi$ is the curvature matrix of the metric $d\sigma^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2$. With this in mind, set

$$(d\chi + \chi \wedge \chi) = \big(\tfrac{1}{2}C_{ijk\ell}\omega^k \wedge \omega^\ell\big)$$

where $\mathrm{Riem}(d\sigma^2) = \frac{1}{4}C_{ijk\ell}(\omega^i \wedge \omega^j) \circ (\omega^k \wedge \omega^\ell)$. (Since ω is orthonormal for $d\sigma^2$ it is immaterial whether indices are written up or down in this formula.) To understand the remaining terms, set

$$\varphi^i_{jk} = dF^i_{jk} - F^i_{j\ell}\chi^\ell_k + F^i_{\ell k}\chi^\ell_j - F^\ell_{jk}\chi^i_\ell\,.$$

Then $\varphi^i_{jk} = F^i_{jk\ell}\omega^\ell$ for some collection of functions $F^i_{jk\ell}$ satisfying

$$\begin{gathered} F^i_{jk\ell} = F^i_{kj\ell} \\ \sum_i F^i_{ik\ell} = \sum_i F^k_{ii\ell} \end{gathered}$$

(Of course, these new functions are just the χ-covariant derivatives of F^i_{jk}.) Then

$$\big(d\varphi + \chi \wedge \varphi + \varphi \wedge \chi\big)^i_j = \varphi^i_{jk} \wedge \omega^k.$$

Substituting this into the above equation yields

$$\left(F^i_{jk\ell}\omega^\ell \wedge \omega^k\right) + \left(\varphi \wedge \varphi + (d\chi + \chi \wedge \chi) - H\,\omega \wedge {}^t\omega\right)^i_j = 0.$$

Now set

$$\left(\varphi \wedge \varphi + (d\chi + \chi \wedge \chi) - H\,\omega \wedge {}^t\omega\right)^i_j = \tfrac{1}{2} T^i_{jk\ell}\omega^k \wedge \omega^\ell$$

where $T^i_{jk\ell} = -T^i_{j\ell k}$. (Notice that T can be computed explicitly in terms of H, $\mathrm{Riem}(d\sigma^2)$, and φ. This formula will be derived presently.) The above equations yield the relations

$$F^i_{jk\ell} - F^i_{j\ell k} = T^i_{jk\ell}\,.$$

Setting $i = k$, $j = \ell$, and summing yields the fundamental relation

$$\begin{aligned}\sum_{i,j} T^i_{jij} &= \sum_{i,j} F^i_{jij} - F^i_{jji} \\ &= \sum_{i,j} F^i_{ijj} - F^i_{jji} = 0\end{aligned}$$

by the above identities for $F^i_{jk\ell}$. A simple index calculation (left to the reader) reveals that, expressed in terms of the first-order quantities discussed before, this relation becomes

$$\left(\sum_m \left(\left(\sum_i F^m_{ii}\right)^2 - \sum_{i,j} F^i_{mj}F^m_{ji}\right)\right) + C - 2\mathrm{tr}(H) = 0$$

where $C = \sum_{i,j} C_{ijij}$ is the scalar curvature of $d\sigma^2$. This is the promised fourth first-order condition on $ds^2 = {}^t\omega\, G\, \omega$.

This relation takes a slightly more comprehensible form if $\{F^i_{jk}\}$ is decomposed into its O(3)-indecomposable components as follows: Using the linear identities on $\{F^i_{jk}\}$ derived above, F^i_{jk} can be expressed uniquely in the form

$$F^i_{jk} = (\delta^i_j a_k + \delta^j_k a_i + \delta^k_i a_j) + (\epsilon^{ij\ell} b_{\ell k} + \epsilon^{ik\ell} b_{\ell j}) + c_{ijk}$$

where δ^i_j is the Kronecker δ, ϵ^{ijk} is totally skew and satisfies $\epsilon^{123} = 1$, $\{a_k\}$ has 3 components, $\{b_{ij} = b_{ji}\}$ has 5 components (it is traceless and symmetric), and $\{c_{ijk}\}$ has 7 components (it is traceless ($\sum_i c_{iik} = 0$) and symmetric). Defining the quadratic forms

$$\begin{aligned}Q_3(a) &= a_1^2 + a_2^2 + a_3^2 \\ Q_5(b) &= \sum_{i,j} (b_{ij})^2 \\ Q^7(c) &= \sum i, j, k (c_{ijk})^2\end{aligned}$$

then the relation above becomes

$$C = 2\mathrm{tr}(H) - 10\, Q_3(a) - 3\, Q_5(b) + Q_7(c).$$

It is important to note that, as an equation for (a, b, c), this relation is non-degenerate for $(a, b, c) \neq (0, 0, 0)$. It is by no means obvious, *a priori*, that one cannot derive more first-order relations by differentiating still further and doing clever index juggling. However, I am now going to show that this cannot be done.

Let $X \subset U \times \mathbb{R}^{21}$ denote the set of quintuples $(m, H, a, b, c) \in U \times \mathbb{R}^6 \times \mathbb{R}^3 \times \mathbb{R}^5 \times \mathbb{R}^7$ which satisfy the relation

$$C(m) = 2\mathrm{tr}(H) - 10\, Q_3(a) - 3\, Q_5(b) + Q_7(c)$$

and for which $(a, c) \neq (0, 0) \in \mathbb{R}^3 \times \mathbb{R}^7$. This X is a smooth manifold which is a hypersurface in $U \times \mathbb{R}^{21}$. Let $\rho_d : \mathrm{O}(3) \to \mathrm{O}(d)$ denote the standard representations of $\mathrm{O}(3)$ for $d = 3$, 5, and 7. For any $g\colon U \to \mathrm{O}(3)$, there is an action

$$(m, H, a, b, c)^g = \bigl(m, g^{-1}\,H\,{}^t g^{-1},\ \rho_3(g^{-1})a,\ \rho_5(g^{-1})b,\ \rho_7(g^{-1})c\bigr)$$

which clearly preserves X. Now, on X^{23}, define the functions and forms

$$\begin{aligned}
F^i_{jk} &= (\delta^i_j a_k + \delta^j_k a_i + \delta^k_i a_j) + (\epsilon^{ij\ell} b_{\ell k} + \epsilon^{ik\ell} b_{\ell j}) + c_{ijk} \\
\gamma &= (\gamma^i_j) = (\chi^i_j + F^i_{jk}\omega^k) \\
\theta &= (\theta^{ij}) = dH + \gamma\,H + H\,{}^t\gamma \\
\Theta &= d\gamma + \gamma \wedge \gamma - H\,\omega \wedge {}^t\omega.
\end{aligned}$$

Also, note the identities

$$\begin{aligned}
d\omega &= -\gamma \wedge \omega \\
\Theta \wedge \omega &= 0 \\
d\theta &= -\gamma \wedge \theta + \theta \wedge {}^t\gamma + \Theta\,H + H\,{}^t\Theta \\
d\Theta &= -\Theta\,\omega \wedge {}^t\omega + \Theta\gamma - \gamma \wedge \Theta.
\end{aligned}$$

Finally, there is an extra identity (which, the reader will find, is equivalent to the relation $C = 2\mathrm{tr}(H) - 10Q_3(a) - 3Q_5(b) + Q_7(c)$), namely

$$\mathrm{tr}([\omega]\Theta) \equiv 0$$

where the notation $[\omega]$ is the same as in §C.

Now let $\mathcal{I}$ denote the differential system generated by the components of θ and Θ. For independence condition take, as usual, $L = span\{\omega^1, \omega^2, \omega^3\}$.

Theorem 1: *The system $(\mathcal{I}, L)$ is in linear form, invariant under change of frame $g\colon U \to \mathrm{O}(3)$, and involutive, with Cartan characters $(s_0, s_1, s_2, s_3) = (6, 9, 5, 0)$. In particular, if $\mathcal{K} \in \Gamma\bigl(S^2(\Lambda^2 T^*)\bigr)$ is real-analytic and positive definite, then the equation $Riem(ds^2) = \mathcal{K}$ is locally solvable.*

PROOF: That $(\mathcal{I}, L)$ is in linear form is obvious. The identities for $d\theta$ and $d\Theta$ show that $\mathcal{I}$ is differentially closed. For any $g\colon U \to \mathrm{O}(3)$ it is easy to compute the formulae

$$\begin{aligned}
\theta^g &= g^{-1}\theta\,{}^t g^{-1} \\
\Theta^g &= g^{-1}\Theta\,{}^t g^{-1}
\end{aligned}$$

so the invariance is obvious also.

In order to prove that the system is involutive, it will be necessary to compute the reduced characters. Since, in degree one, $\mathcal{I}$ is generated by the 6 components of θ, it follows that $s_0' = 6$. Set

$$\varphi^i_{jk} = dF^i_{jk} - F^i_{j\ell}\chi^\ell_k - F^i_{\ell k}\chi^\ell_j + F^\ell_{jk}\chi^i_\ell$$

and note that $\Theta^i_j = \varphi^i_{jk} \wedge \omega^k +$ (terms quadratic in $\omega^1, \omega^2, \omega^3$).

Now the φ^i_{jk} are linearly independent modulo θ, ω *except* for the following relations

$$\varphi^i_{jk} = \varphi^i_{kj}$$

$$\sum_i \varphi^k_{ii} = \sum_i \varphi^i_{ik}$$

$$\sum_m \left(2 \left(\sum_i F^m_{ii} \right) \left(\sum_j \varphi^m_{jj} \right) - \sum_{i,j} \left(F^i_{jm} \varphi^m_{ji} + F^m_{ji} \varphi^i_{jm} \right) \right) \equiv \sum_i \theta^{ii} \bmod \omega.$$

The reduced polar spaces relative to the reduced flag specified by $(\omega^1, \omega^2, \omega^3)$ are given in the following table

Reduced Flag Definition	**Reduced Polar Equations**
$\omega^1 = \omega^2 = \omega^3 = 0$	$\theta^{ij} = 0$
$\omega^2 = \omega^3 = 0$	$\varphi^i_{j1} = \theta^{ij} = 0$
$\omega^3 = 0$	$\varphi^i_{j2} = \varphi^i_{j1} = \theta^{ij} = 0$
	$\varphi^i_{jk} = \theta^{ij} = 0$

From this, one immediately sees that $s'_0 = 6$, $s'_1 = 9$, and $s'_0 + s'_1 + s'_2 + s'_3 = 20$. Note that, by symmetry, the only terms in $\{\varphi^i_{jk}, \theta^{ij}\}$ which could fail to appear in $\{\varphi^i_{j1}, \varphi^i_{j2}, \theta^{ij}\}$ are the three terms $\{\varphi^1_{33}, \varphi^2_{33}, \varphi^3_{33}\}$. On the other hand, the second relations on φ^i_{jk} show that $\{\varphi^1_{33}, \varphi^2_{33}\}$ are linear combinations of $\{\varphi^i_{j1}, \varphi^i_{j2}\}$. Thus, $s'_0 + s'_1 + s'_2 \geq 19$ and will equal 20 iff the last relation on the φ^i_{jk} allows one to solve for φ^3_{33} in terms of the remaining forms. However, the coefficient of φ^3_{33} in this last expression is easily seen to be $2(F^3_{11} + F^3_{22}) = 2(2a_3 - c_{333})$. This latter expression vanishes in all coframings $\omega^g = g^{-1}\omega$ iff $a = c = 0$ since a and c belong to different representations of O(3). Since, by construction, $(a, c) \neq (0, 0)$, it follows that there is always a frame so that $2a_3 - c_{333} \neq 0$. Hence, in such a frame, $s'_0 + s'_1 + s'_2 = 20$. This, together with the other relations derived above, establishes the reduced character sequence $(s'_0, s'_1, s'_2, s'_3) = (6, 9, 5, 0)$.

It remains to apply Cartan's Test. An admissible element is defined by relations of the form $\theta^{ij} = r^{ij}_k \omega^k$ and $\varphi^i_{jk} = s^i_{jk\ell} \omega^\ell$ where $r^{ij}_k = r^{ji}_k$ and $s^i_{jk\ell}$ satisfy the conditions

$$s^i_{jk\ell} = s^i_{kj\ell}$$
$$\sum_i s^k_{ii\ell} = \sum_i s^i_{ik\ell}$$

and three more conditions given by the remaining relation on the φ^i_{jk}. These relations merely ensure that the relations $\theta^{ij} - r^{ij}_k \omega^k = 0$ and $\varphi^i_{jk} - s^i_{jk\ell}\omega^\ell = 0$ actually define an admissible element in X. An admissible element must, in addition, satisfy $r^{ij} = 0$ in order that $\theta^{ij} = 0$. This is $6 \times 3 = 18$ conditions. Then, the $s^i_{jk\ell}$ must satisfy the conditions which annihilate the 9 2-forms Θ^i. This would be 27 conditions, but the relations $\Theta{\scriptstyle\wedge}\omega = \mathrm{tr}([\omega]\Theta) = 0$ show that 4 of these conditions are identities. Thus, an admissible element need only satisfy at most $18 + 23 = 41 = N$ conditions. Since

$$N = 41 \leq 3 \cdot 6 + 2 \cdot 9 + 5,$$

it follows that Cartan's test succeeds and the system is involutive. ☐

The above argument computes the characteristic variety as well. Let $P(\xi_1, \xi_2, \xi_3)$ be the polynomial on $X \times_U T^*U$ given by

$$P = 2\big(\sum_i a_i \xi_i\big)\big(\xi_1^2 + \xi_2^2 + \xi_3^2\big) - \sum_{i,j,k} c_{ijk}\xi_i\xi_j\xi_k\,.$$

Then it is easy to see that $\xi = \xi_i\omega^i$ is characteristic iff either $\xi_1^2 + \xi_2^2 + \xi_3^2 = 0$ or $P(\xi_1, \xi_2, \xi_3) = 0$. Thus the characteristic variety at each point is the union of a quadric and a cubic. In particular, the *real* characteristic variety is never empty so the system is never elliptic. It is also worth remarking that the characteristics do *not* depend on the value of H at a point. This is in striking contrast to the prescribed Ricci equation [D] (where the characteristics at a point depend *only* on the value of H). Of course, similar results hold if $\mathcal{K}$ has other signatures: (2,1), (1,2), (0,3).

It may be worth remarking on the nature of the identities uncovered in the course of this discussion. Let $\mathcal{K} \in \Gamma\big(S^2(\Lambda^2 T^*M)\big)$ be of non-degenerate type and let $\Sigma_k \subset J^k(S^2_+(T^*))$ denote the subset of k-jets of solutions of $\mathrm{Riem}(ds^2) = \mathcal{K}$ for $k \geq 2$. The Bianchi identities state that the natural projection $\Sigma_3 \to J^1\big(S^2_+(T^*)\big)$ is *not* surjective (even though $\Sigma_2 \to J^1\big(S^2_+(T^*)\big)$ is clearly surjective) and that, moreover, its image is a smooth codimension 3 submanifold $B_1 \subset J^1\big(S^2_+(T^*)\big)$ (consisting of the 1-jets satisfying the second Bianchi identity). The "new" identity above shows that the map $\Sigma_4 \to B_1 \subset J^1$ is not surjective either. Not only that, but the image $B_2 \subset B_1$ need not be smooth. However, away from a small singular set, B_2 *is* a smooth hypersurface in B_1. Let $B_2^* \subset B_2$ denote this smooth part of the hypersurface. Then it is a consequence of Theorem 1 that, for all $k \geq 4$, $\Sigma_k \to J^1$ covers B_2^*. Moreover, letting $\Sigma_k^* \subset \Sigma_k$ denote the inverse image of B_2^* for all $k \geq 2$, the maps $\Sigma_k \to \Sigma_{k-1}^*$ are smooth submersions for all $k \geq 3$ while $\Sigma_2^* \to B_2^*$ is a smooth submersion. Thus, there are no more "hidden" Bianchi relations no matter how far one differentiates.

Making use of the formal solvability result contained in the above theorem, DeTurck and Yang have shown that $\mathrm{Riem}(ds^2) = \mathcal{K}$ is solvable in the smooth category for non-degenerate $\mathcal{K}$, see [DY].