Exercises for Day 7: Tangent bundle and submanifolds

1. Stereographic projection. Show that $S^n \subset \mathbb{R}^{n+1}$ (with it's manifold structure inherited as a submanifold of $\mathbb{R}^{n+1}$) can be covered by an atlas consisting of two charts $(V_\pm, \phi_\pm)$ where $V_\pm = S^n \setminus \{(0,\dots,0,\pm1)\}$ and

$$\phi_\pm(x^1,\dots,x^{n+1}) = \left(\frac{x^1}{1\mp x^{n+1}},\dots,\frac{x^n}{1\mp x^{n+1}}\right).$$

2. The Riemann Sphere. In the special case of S^2, we often make a different choice for $\phi_- : V_- \to \mathbb{R}^2$, instead using

$$\phi_-(x^1,x^2,x^3) = \left(\frac{x^1}{1+x^3},\frac{-x^2}{1+x^3}\right).$$

The reason is that, with this choice, one can compute that

$$\phi_+\circ(\phi_-)^{-1}(y^1,y^2) = \left(\frac{y^1}{(y^1)^2+(y^2)^2},\frac{-y^2}{(y^1)^2+(y^2)^2}\right).$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$ via $(y^1,y^2) = y^1+\sqrt{-1}\,y^2$, then this map just becomes $\phi_+\circ(\phi_-)^{-1}(z) = \bar z/(z\bar z) = 1/z$, so that the transition map is *holomorphic* (i.e., complex differentiable), not just smooth. Using this pair of charts to make an atlas, we can make sense of what it means for a complex-valued function f on S^2 to be holomorphic or for a mapping $f : S^2 \to S^2$ to be holomorphic. When S^2 is endowed with this pair of charts, we say that it is a *holomorphic manifold* (of complex dimension 1). It is known as the *Riemann sphere*. We usually identify $z \in \mathbb{C}$ with $(\phi_+)^{-1}(z) \in V_+ \subset S^2$ and call the 'missing point', i.e., $(0,0,1)$, the 'point at $z=\infty$'.

Here is a simple application of this structure: Show that, if $p(z) = p_n z^n + p_{n-1}z^{n-1}+\cdots+p_0$ is a complex polynomial of degree $n > 0$ (i.e., $p_n \neq 0$), then we can define a nonconstant holomorphic mapping $f : S^2 \to S^2$ by the rule $f(0,0,1) = (0,0,1)$ and $f\circ(\phi_+)^{-1}(z) = (\phi_+)^{-1}\big(p(z)\big)$ for $z \in \mathbb{C} = \mathbb{R}^2$. Using the fact that a nonconstant holomorphic mapping is open, conclude that the image $f(S^2)$ is open in S^2. Since S^2 is connected and $f(S^2)$ must be compact (and hence closed), it follows that $f(S^2) = S^2$. Conclude that there must be a point $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$. (This result is known as the Fundamental Theorem of Algebra.)

3. The tangent bundle of S^3. Consider the three matrices

$$I = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix},\quad J = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix},\quad K = \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}.$$

Show that, for any $x \in S^3$, the tangent space $T_xS^3 \subset \mathbb{R}^4$ is spanned by the three vectors Ix, Jx, and Kx. Conclude that there is a diffeomorphism $TS^3 \simeq S^3 \times \mathbb{R}^3$. (We will eventually see that, for example, $TS^2 \not\simeq S^2\times\mathbb{R}^2$.)

4. The Hopf map $\pi : S^3 \to S^2$. Show that the formula

$$\pi(x^1,x^2,x^3,x^4) = \big((x^1)^2+(x^2)^2-(x^3)^2-(x^4)^2,\ 2(x^1x^3+x^2x^4),\ 2(x^1x^4-x^2x^3)\big)$$

defines a smooth submersion $\pi : S^3 \to S^2$. Show that the kernel of $\pi'(x)$ is spanned by Ix for all $x \in S^3$.

5. Show that if $f : X \to Y$ is a smooth submersion, where X is compact and Y is connected, then f is onto, i.e., $f(X) = Y$. (Hint: Show that a submersion is an open mapping.)

6. Show that, for $A \in O(n)$, we have $T_AO(n) = \{As \mid s + {}^ts = 0\} \subset M_{n,n}(\mathbb{R})$. Show also that, if $\pi : O(n) \to S^{n-1}$ is defined so that $\pi(a)$ is the first column of a, then π is a submersion and that $\pi^{-1}\big(\pi(a)\big) = a\cdot O(n{-}1)$, where $O(n{-}1) \subset O(n)$ is the subgroup whose first column is ${}^t(1,0,\dots,0)$.