Exercises for Day 5: Tangent vectors and tangent planes

1. Show that, if $M \subset \mathbb{R}^n$ is an embedded smooth submanifold of dimension m and $x \in M$ is fixed, then T_xM can be naturally identified with the set of velocities $c'(0)$ (in the usual sense) of curves $c : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ that lie in M and satisfy $c(0) = x$. In particular, show that this set of velocities is a linear subspace of $\mathbb{R}^n$. (From now on, we will identify T_xM with this subspace of $\mathbb{R}^n$ unless there is some possibility of confusion.)

2. Show that, for $x \in S^n \subset \mathbb{R}^n$, the vector space T_xS^n can be identified with the subspace

$$x^\perp = \{v \in \mathbb{R}^n \bigm| x \cdot v = 0\}.$$

3. More generally, show that if $D \subset \mathbb{R}^m$ is an open set, $f : D \to \mathbb{R}^n$ is a smooth mapping, and $b \in \mathbb{R}^n$ is a regular value for f, then, for all $x \in f^{-1}(b) = M$, we have $T_xM = \ker f'(x)$.

4. Let M be a smooth manifold, $U \subset M$ an open set, and $K \subset U$ a compact subset. Show that there exists an open set $U' \subset U$ that contains K and is such that, for any smooth function $f : U \to \mathbb{R}$, there exists a smooth function $\tilde{f} : M \to \mathbb{R}$ such that $\tilde{f}$ agrees with f on U'. (Hint: It suffices to find a smooth function $b : M \to \mathbb{R}$ such that b is identically equal to 1 on an open neighborhood of K but such that the support[1] of b is a compact subset of U. To make such a function, cover K with open balls lying in U on which such "bump" functions have been defined and then figure out how to combine them.)

5. Alternate definition of T_xM. Let M be a smooth n-manifold with atlas $\mathcal{A}$. Let S be the set of triples $\bigl(x, (V, \phi), v\bigr) \in M \times \mathcal{A} \times \mathbb{R}^n$ with $x \in V$. We define an equivalence relation on S by saying that $\bigl(x, (U, \phi), v\bigr) \sim \bigl(y, (V, \psi), w\bigr)$ if and only if $x = y$ and $\bigl(\psi\circ\phi^{-1}\bigr)'\bigl(\phi(x)\bigr)(v) = w$. (Note that this last equation makes sense because $\psi\circ\phi^{-1}$ is a smooth map from an open neighborhood of $\phi(x) \in \mathbb{R}^n$ to $\mathbb{R}^n$. We define TM to be the set of $\sim$-equivalence classes in S and we define $T_xM \subset TM$ to be the set of equivalence classes of the elements $\bigl(x, (V, \phi), v\bigr)$.

a. Verify that $\sim$ is indeed an equivalence relation.

b. Let $c : (-\varepsilon, \varepsilon) \to M$ be a smooth map with $c(0) = x$. Show that, if we assign $c'(0)$ to the $\sim$-equivalence class of $\bigl(x, (V, \phi), (\phi\circ c)'(0)\bigr)$, then this assignment defines a bijection from T_xM as defined in class with T_xM as defined in this exercise.

c. Show that, if we fix a chart $(V, \phi) \in \mathcal{A}$ with $x \in V$, then every element of T_xM has a representative of the form $\bigl(x, (V, \phi), v\bigr)$ for some unique $v \in \mathbb{R}^n$. Thus, T_xM has an identification with $\mathbb{R}^n$ (which depends on the choice of chart (V, ϕ)). Finally, show that there is a unique vector space structure on T_xM so that each of these identifications is a linear isomorphism.

6. Show that, if M and N are two smooth manifolds, then $T_{(x,y)}(M \times N)$ is canonically isomorphic to the vector space sum $T_xM \oplus T_yN$.

[1] The *support* of a function $f : M \to \mathbb{R}$ is the closure of the open set where f is nonzero.