

Exercises for Day 12: The Exponential Map, I

In these exercises g will denote a Riemannian metric on M . At each point $p \in M$, there is a unique inner product $\langle \cdot, \cdot \rangle_p$ on $T_p M$ such that $\langle v, v \rangle_p = g(v)$ for all $v \in T_p M$. When the notation $\langle v, w \rangle$ is used, it is intended that v and w lie in $T_p M$ for some $p \in M$.

1. GEODESIC SPEEDS. Show that any geodesic $\gamma : (a, b) \rightarrow M$ has constant speed, i.e., $|\gamma'(t)| = c$ for some constant c . (Hint: It suffices to check this in coordinates.)

Also, verify that, if $\gamma : (a, b) \rightarrow M$ is a geodesic, then for any real number $r \neq 0$, the curve $\tilde{\gamma} : r \cdot (a, b) \rightarrow M$ defined by $\tilde{\gamma}(t) = \gamma(t/r)$ is also a geodesic. Conclude that, for any $v \in TM$ with $I_v \subset \mathbb{R}$ being the domain of the flow of the Euler vector field, we have that $I_{rv} = r^{-1} \cdot I_v$ for all $r \neq 0$.

Finally, verify that, if $\gamma : (a, b) \rightarrow M$ is a maximally extended geodesic (i.e., it cannot be defined on any larger domain in \mathbb{R}), then, for any $c \in (a, b)$, then the curve $\tilde{\gamma} : (a-c, b-c) \rightarrow M$ defined by $\tilde{\gamma}(t) = \gamma(t+c)$ is the maximally extended geodesic that satisfies $\tilde{\gamma}'(0) = \gamma'(c)$.

2. THE EXPONENTIAL MAP ON THE n -SPHERE. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit n -sphere. For any $p \in S^n$, we have $T_p S^n = p^\perp = \{v \in \mathbb{R}^{n+1} \mid p \cdot v = 0\}$. The object of this exercise is to compute the exponential map $\exp : TS^n \rightarrow S^n$.

First, explain why, if (M, g) is a Riemannian manifold and $f : M \rightarrow M$ is an isometry of g that satisfies $f'(x)(v) = v$ for some $v \in T_x M$, then, for the geodesic $\gamma : (a, b) \rightarrow M$ with $\gamma'(0) = v$, we have $f \circ \gamma = \gamma$.

Next, show that, if $(p, v) \in TS^n$ (i.e., $p \cdot v = 0$) has $v \neq 0$, then the reflection $R : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ that fixes pointwise the plane spanned by p and v and reverses all vectors perpendicular to p and v is an isometry of S^n . Explain why this shows that the geodesic $\gamma : (a, b) \rightarrow S^n \subset \mathbb{R}^{n+1}$ that satisfies $\gamma(0) = p$ and $\gamma'(0) = v$ must be of the form

$$\gamma(t) = c(t)p + s(t)\frac{v}{|v|}$$

for some smooth functions c and s . Explain why we must have

$$c(t) = \cos(t|v|) \quad \text{and} \quad s(t) = \sin(t|v|).$$

(Hint: Use the fact that $|\gamma(t)| \equiv 1$ and $|\gamma'(t)| = |v|$.)

3. EXTENDING GEODESICS. Show that, if $K \subset M$ is compact then there is a $\delta > 0$ such that, for any $p \in K$ and any $v \in T_p M$ with $|v| \leq 1$, the geodesic with initial velocity v can be extended at least over the interval $(-\delta, \delta)$.

Next, show that, if $\gamma : (a, b) \rightarrow M$ is a maximally extended geodesic such that $b < \infty$, then γ 'eventually leaves any compact set in M ', i.e., if $K \subset M$ is compact, then there is a $\tau \in (a, b)$ such that $\gamma(t) \notin K$ for all $t \in (\tau, b)$. (Hint: If not, then there is a sequence of times $t_i \in (a, b)$ such that t_i converges to b but $\gamma(t_i)$ lies in K . Why can this not happen?)

4. GEODESICS FROM THE EXPONENTIAL MAP. If $\Phi_E : U_E \rightarrow TM$ is the flow of the exponential vector field, show that $\Phi(t, rv) = \Phi(rt, v)$ for all real numbers r for which either side makes sense. Conclude that, for any vector $v \in TM$, the curve $\gamma_v(t) = \exp(tv)$ for $t \in I_v$ is the geodesic with initial velocity v .