Math 104X   Second In-Class Exam   8 March 2007

This examination consists of 3 double-sided pages. You have 75 minutes to complete it. **Be sure to sign the Honor Code Acknowledgement on the last page.**

Do all of the work you want graded on the pages of this exam. Be as neat as possible, since I will ignore anything that I cannot read. Good luck.

10 points **1**. Circle the letters of the statements that are true. In each case, give a *brief* justification for your answer or a counterexample.

a. $\det(A+B) = \det(A) + \det(B)$.

False. $\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \neq \det\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \det\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$.

b. If G is symmetric and n-by-n, then the formula $(x, y) = x^T G y$ defines an inner product on $\mathbb{R}^n$.

True. Because $G = G^T$, we have $(x, y) = x^T G y = y^T G^T x = y^T G x = (y, x)$. The other two properties, $(x, y_1 + y_2) = (x, y_1) + (x, y_2)$ and $(x, \lambda y) = \lambda(x, y)$, are consequences of matrix multiplication.

c. $\det(AB) = \det(BA)$ if A and B are any n-by-n matrices.

True. $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$.

d. If $A = -A^T$, then $\det(A) = 0$.

False. $A = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = -A^T$ has $\det(A) = 1$. (It *is* true if the size of A is odd.)

e. If $P \neq I$ is an n-by-n projection matrix, then $\det(P) = 0$.

True. If P is a projection matrix that is not the identity, then it is a projection onto a proper subspace and hence has a nonzero kernel and is not invertible. Thus, $\det(P) = 0$.

5 points **2**. Let A be an n-by-n matrix with integer entries. Explain why $\det(A) = \pm 1$ if and only if A is invertible and A^{-1} also has integer entries.

If $\det(A) = \pm 1$, then A is invertible and, since $A^{-1} = (\det A)^{-1} C^T = \pm C^T$ while the entries of C (the cofactor matrix) are, up to sign, determinants of minors of A, which are also integer matrices and so have integer determinants.

If A is invertible and A^{-1} has integer entries, then both $\det(A)$ and $\det(A^{-1}) = 1/\det(A)$ are integers. However, the only integers whose inverses are also integers are 1 and -1.

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10 points **3**. Compute the determinant of each of the following matrices

$$(i)\ \ A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 \\ 6 & 7 & 8 & 9 \end{vmatrix} = (-1)\begin{vmatrix} 2 & 3 & 4 \\ 0 & 1 & 0 \\ 6 & 7 & 8 \end{vmatrix} = (-1)(1)\begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix} = (-1)(2 \cdot 8 - 6 \cdot 4) = 8.$$

$$(ii)\ \ B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

$$\det B = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$
$$= 1$$

5 points **4**. Apply the Gramm-Schmidt process to the vectors

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix},$$

to produce an orthonormal basis (u_1, u_2, u_3) of $\mathbb{R}^3$.

(I'll spare you the arithmetic and just give the answer.)

$$u_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad u_2 = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad u_3 = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

10 points **5**. Find the line $y = ax+b$ that best approximates the data $(x, y) = (0, 1)$, $(1, 3)$, and $(2, 3)$ in the sense of least squares.

The desired line $y = ax + b$ will be found by finding the best 'approximate solution' to the (incompatible) system

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}.$$

The corresponding normal system is

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix},$$

so the line is $y = x + \frac{4}{3}$.

10 points **6**. Find the 3-by-3 matrix A that represents a reflection in the plane $x_1 + 2x_2 - 2x_3 = 0$. (Hint: First, find the projection onto the orthogonal line.)

The projection matrix is given by

$$P = \frac{1}{9} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \end{pmatrix} = \frac{1}{9} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}.$$

The reflection matrix is then

$$R = I - 2P = \frac{1}{9} \begin{bmatrix} 7 & -4 & 4 \\ -4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}.$$

10 points **7**. Let V be a vector space with basis $v_1, \ldots, v_n$ and let $(\,,)$ be an inner product on V. Define the matrix $G = (G_{ij})$ by

$$G_{ij} = (v_i, v_j).$$

Explain why G is symmetric. If

$$x = x_1\, v_1 + \cdots + x_n\, v_n, \qquad y = y_1\, v_1 + \cdots + y_n\, v_n,$$

show that

$$(x, y) = \sum_{i,j=1}^{n} G_{ij} x_i y_j$$

and explain why $(\,,)$ is nondegenerate if and only if G is invertible. What is the relationship between the kernel of G and the subspace $V^\perp = \{\, y \in V \,\big|\, (x, y) = 0 \text{ for all } x \in V \,\}$?

Since $G_{ji} = (v_j, v_i) = (v_i, v_j) = G_{ij}$, it is clear that G is symmetric. Now, using the above formulae for x and y above and the linearity of $(\,,)$ in both arguments, we see

$$(x, y) = \left(\sum_{i=1}^{n} x_i v_i,\ \sum_{i=1}^{n} y_i v_i \right) = \sum_{i,j=1}^{n} x_i y_j (v_i, v_j) = \sum_{i,j=1}^{n} G_{ij} x_i y_j\,.$$

If G is not invertible, then there are numbers $y_1, \ldots, y_n$, not all zero, such that $\sum_{j=1}^{n} G_{ij} y_j = 0$ for all i. But then the vector $y = y_1\, v_1 + \cdots + y_n\, v_n$ will be nonzero and satisfy $(x, y) = 0$ for all $x \in V$, so $(\,,)$ will be degenerate. Conversely, if $(\,,)$ is nondegenerate, then, for any numbers $y_1, \ldots, y_n$, not all zero, the vector $y = y_1\, v_1 + \cdots + y_n\, v_n$ in V is nonzero, so there will be an $x = x_1\, v_1 + \cdots + x_n\, v_n$ in V such that $(x, y) \neq 0$. By the above formula, it then follows that we cannot have $\sum_{j=1}^{n} G_{ij} y_j = 0$ for all i. Thus G has no kernel and hence is invertible.

By the discussion in the previous paragraph, it follows that $y = y_1\, v_1 + \cdots + y_n\, v_n$ lies in $V^\perp$ if and only if the vector $(y_1, \ldots, y_n)$ lies in the kernel of G. In fact, the linear map $F : \mathbb{R}^n \to V$ given by $F(y_1, \ldots, y_n) = y_1\, v_1 + \cdots + y_n\, v_n$, which is one-to-one and onto because $v_1, \ldots, v_n$ is a basis of V, carries the kernel of G onto $V^\perp$. In particular, the dimension of $V^\perp$ is $n - r$, where r is the rank of G.

10 points **8**. The point of this problem is to show that, if A is an n-by-n matrix that satisfies $A^2 = 0$, then $\det(I_n + A) = 1$. Let $p(t) = \det(I_n + tA)$. Supply justifications for the following statments:

(*i*) $p(t)$ is a polynomial in t of degree at most n.

By our formula for the determinant, $p(t)$ is a sum of $n!$ terms, each of which is a product of n entries from $(I_n + tA)$. Since each of these factors is linear in t, the product has degree at most n in t.

(*ii*) $p(t)^2 = p(2t)$.

Since $A^2 = 0$,

$$\begin{aligned} p(t)^2 &= \det(I_n + tA)^2 = \det\big((I_n + tA)^2\big) = \det\big((I_n + 2tA + t^2A^2)\big) \\ &= \det\big((I_n + 2tA)\big) = p(2t). \end{aligned}$$

(*iii*) p has degree 0.

If $p(t)$ has degree d in t, then $p(t)^2$ will have degree $2d$ in t while $p(2t)$ will still have degree d in t. Thus $2d = d$, i.e., $d = 0$.

(*iv*) $p(1) = p(0)$.

Since $p(t)$ has degree 0 in t, it must be constant. In particular, $p(t) = p(0)$ for all t.

(*v*) $\det(I_n + A) = 1$.

$\det(I_n + A) = p(1) = p(0) = \det(I_n + 0) = 1$.

Remark: This argument can be generalized to show that $\det(I_n + A) = 1$ for any matrix A that satisfies $A^{k+1} = 0$ for some $k \geq 0$. The point is that the polynomial $p(t) = \det(I_n + tA)$, which has some degree $d \leq n$ in t, has the property that, for any $m \geq 1$, the power $p(t)^m$, which has degree md, also has an expression as a polynomial in t of degree at most kn, since, by the binomial theorem and the assumption $A^{k+1} = 0$,

$$p(t)^m = \det\left((I_n + tA)^m\right) = \det\left(I_n + mtA + \binom{m}{2}t^2A^2 + \cdots + \binom{m}{k}t^kA^k\right).$$

If $d > 0$, then as soon as you take $m > kn/d$, you get a contradiction.

Remark: Also, the equation $p(t)^2 = p(2t)$, by itself, does not force p to be a constant function. Consider $p(t) = e^t$.

10 points **9**. Let $(\,,\,)$ be an inner product on the vector space V. For any $v \in V$, the map $f : V \to \mathbb{R}$ defined by $f(x) = (v, x)$ is linear. The point of this problem is to prove a (partial) converse: If $(\,,\,)$ is nondegenerate and V is finite dimensional, then any linear map $f : V \to \mathbb{R}$ can be written as $f(x) = (v, x)$ for some unique $v \in V$. Supply explanations and/or descriptions as required:

(i) Let $e_1, \ldots, e_n$ be a basis of V and define the linear map $F : V \to \mathbb{R}^n$ by

$$F(x) = \big((e_1, x), (e_2, x), \ldots, (e_n, x)\big).$$

Describe the kernel and range of F. (In particular, why is nondegeneracy important?)

The kernel of F is the zero subspace and the range of F is all of $\mathbb{R}^n$. The reason is as follows: If $F(x) = 0$, then x is perpendicular to each of the vectors $e_1, \ldots, e_n$ and hence is perpendicular to their span, which is all of V. In other words $x \in V^\perp$, but since $(\,,\,)$ is nondegenerate, $V^\perp = 0$. Thus, $x = 0$. This shows that the kernel of F is the zero subspace. By the rank equation,

$$n = \dim V = \dim \ker(F) + \dim \operatorname{range}(F) = 0 + \dim \operatorname{range}(F),$$

so $\operatorname{range}(F) \subseteq \mathbb{R}^n$ has dimension n and so $\operatorname{range}(F) = \mathbb{R}^n$, as claimed.

(ii) Explain why there is a unique vector $v \in V$ such that

$$F(v) = \big(f(e_1), f(e_2), \ldots, f(e_n)\big).$$

Because F is one-to-one (since $\ker(F) = 0$) and onto (since $\operatorname{range}(F) = 0$), there is only one vector $v \in V$ that gets sent to $\big(f(e_1), f(e_2), \ldots, f(e_n)\big)$ by F.

(iii) Explain why $f(x) = (v, x)$ for all $x \in V$.

Let $v \in V$ be the unique solution to the above equation. Then $f(e_i) = (v, e_i)$ for $i = 1, \ldots, n$. Any vector $x \in V$ can be written uniquely in the form $x = x_1\, e_1 + \cdots + x_n\, e_n$ and hence

$$\begin{aligned} f(x) &= f(x_1\, e_1 + \cdots + x_n\, e_n) = x_1\, f(e_1) + \cdots + x_n\, f(e_n) \\ &= x_1\,(v, e_1) + \cdots + x_n\,(v, e_n) = (v, x_1\, e_1 + \cdots + x_n\, e_n) = (v, x). \end{aligned}$$

(Note that v is unique because $(\,,\,)$ is nondegenerate.)

Honor Code Statement: I have neither given nor received nor will I provide unacknowledged aid on this examination.

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