# Research accomplishement and plans

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My main mathematical interest is at the interface of algebraic geometry, differential geometry, and mathematical physics. At this frontier one finds vector bundles, equipped with special structures such as connections, hermitian metrics, holomorphic structures, special endomorphisms, etc.

There is a dynamical exchange offered by diverse correspondences between the algebraic and differential world, offering the possibility of using the tools of one domain to understand the other (Nahm, Fourier–Mukai, Atiyah–Ward, Kobayashi–Hitchin, etc.). This phenomenon is very well illustrated in the recent work of Kapustin–Witten on the geometric Langlands program where tools of gauge theory (differential world), inspired by physics, are used to understand the algebraic geometry.

For those of you that prefer to read only one page, let me start with a quick outline of my research.

1. **Accomplishments in geometry** page 2
   - The proof that the Nahm transform for calorons (instantons on $S^1 \times \mathbb{R}^3$) is an involution was completed by Hurtubise and me in [CH09].
   - In [CH08b], we gave a description in terms of matrices of the rank 2 calorons. This result allowed in particular to prove that the moduli spaces are non-empty for all charges.
   - Partial understanding of the Nahm transform on $\mathbb{R} \times T^3$ comes from my thesis work [Cha04, Cha06].
   - We proved in [CH08a] a correspondence (see Theorem 1.5) between irreducible singular monopoles on the product $S^1 \times \Sigma$ of a circle and a Riemann surface (differential world) and stable pairs on $\Sigma$ (algebraic world).
2. **Accomplishments in interdisciplinary work** page 4
   - I contributed geometrically to a paper in statistics.
   - I co-mentored a summer undergraduate project a couple of summers ago. This project lead to a paper about numerical methods for approximating to solutions of stochastic differential equations.
   - My geometric intuition and algebraic skills helped a project to understand geometrical frustration and hard sphere crystallization in statistical mechanics.
3. **Current research** page 5
   - I am working on proving an existence and non-existence result for instantons on $\mathbb{R} \times T^3$.
   - I am working on a study of the moduli spaces of Spin(7) Donaldson–Thomas connections.
4. **Plans for future research** page 6
   - Hurtubise and I plan to build up on our earlier work to complete the proof of the heuristic of the Nahm transform on $\mathbb{R} \times T^3$.
   - Jardim, Moraru and I plan to study the Fourier–Mukai/Nahm transform of meromorphic Higgs bundles.
   - I have many other projects in mind: integrable system structure on the moduli spaces of calorons, caloron counting, monopole sheets moduli, classification of gravitational instantons and Yang–Mills instantons on those, etc.
5. **Training of students** page 7
   - Many questions coming out of the study of the topics itemized above have the correct level of difficulties to be addressed by graduate and undergraduate students.

The next pages expand on these ideas, providing background information and explaining the relevance of my work.

# A one page summary of the background material

Let $X$ be a Riemannian manifold and $E$ be a vector bundle on $X$. The *Yang–Mills equation* $d_\nabla^* F_\nabla$ for a connection $\nabla$ on $E$ of curvature $F_\nabla$ is the basic equation underlying my work. Solutions are critical points of $\int_X |F_\nabla|^2$.

When $\dim X = 4$, minimizers are solutions to the self-dual (SD) or anti-self-dual (ASD) equation $*F_\nabla = \pm F_\nabla$ and are called *instantons*. The connections $\nabla$ satisfy the SD or ASD equation if $F_\nabla \in \bigwedge^\pm \otimes \mathrm{End}(E)$. The decomposition $\bigwedge^2 = \bigwedge^+ \oplus \bigwedge^-$ in irreducible representations of SO(3) arise as the $\pm 1$ eigenspace decomposition for the Hodge star.

When $\dim X = 8$ and holonomy$(X) \subset \mathrm{Spin}(7)$, there is a parallel four-form $\eta$ on $X$. The decomposition $\bigwedge^2 = \bigwedge^2_7 \oplus \bigwedge^2_{21}$ in irreducible representations of Spin(7) arises as the eigenbundles decomposition for $\alpha \mapsto *(\eta \wedge \alpha)$. The connections whose curvature lie in $\bigwedge^2_{21}$ are solutions to the Yang–Mills equation and are called Spin(7)-instantons, or, *Donaldson–Thomas instantons* (DT-instantons) in honor of their introduction in [DT98].

There is a similarly story for any manifold with special holonomy $\mathrm{G}_2$, or holonomy in $\mathrm{SU}(n)$, etc.

When $\dim X = k < 4$, we consider the *dimensional reduction* of the ASD equation:

$$F^+_\nabla = 0 \xrightarrow{\dim = 3} F_\nabla = *d_\nabla \Phi \qquad \text{Bogomolny's equation (solutions are called } monopoles\text{)}$$

$$F^+_\nabla = 0 \xrightarrow{\dim = 2} F_\nabla + [\theta, \theta^*] = 0, \nabla^{0,1}\theta = 0 \qquad \text{Hitchin's equations}$$

$$F^+_\nabla = 0 \xrightarrow{\dim = 1} \nabla_t T_{\sigma(1)} = [T_{\sigma(2)}, T_{\sigma(3)}] \quad (\forall \sigma \text{ even}) \qquad \text{Nahm's equations}$$

The maps $\Phi\colon E \to E$ and $\theta\colon E \to E \otimes K$ ($K$ is the canonical bundle) are called *Higgs fields*.

The *Nahm transform*, in its simplest form, is a heuristic correspondence between

- ♡ solutions to the anti-self-dual (ASD) equation, or its the appropriate dimensional reduction, on the quotient $X$ of $\mathbb{R}^4$ by a closed subgroup $\Lambda$ of $\mathbb{R}^4$, and satisfying a finite energy condition, and
- ♣ solutions to some associate equation satisfying some boundary condition on the quotient $X^*$ of $\mathbb{R}^4$ by the dual subgroup $\Lambda^* = \{f \in \mathbb{R}^4 \mid \langle f, \Lambda\rangle \subset \mathbb{Z}\}$.

The relationship $X \leftrightarrow X^*$ can be seen by transposing the following diagram:

| | | | | | | |
|---|---|---|---|---|---|---|
| | $T^4$ | $\underline{\mathbb{R} \times \mathbf{T^3}}$ | $\mathbb{R}^2 \times T^2$ | $\mathbb{R}^\mathbf{3} \times \mathbf{S^1}$ | $\mathbb{R}^4$ | ⟵ this row: system of PDE |
| | $\underline{\mathbf{T^3}}$ | $\underline{\mathbb{R} \times T^2}$ | $\mathbb{R}^2 \times T^1$ | $\mathbb{R}^3$ | | |
| | $T^2$ | $\mathbb{R} \times S^1$ | $\mathbb{R}^2$ | | | |
| this row: system of ODE ⟶ | $\mathbf{S^1}$ | $\mathbb{R}$ | | | $X$ | |
| this row: algebraic equations ⟶ | $*$ | | | $X^*$ | | |

I have contributed to the investigation of the Nahm transform for those entries in boldface.

In a nutshell, the heuristic starts with a connection $\nabla$ satisfying the SD or ASD equation (or the appropriate dimensional reduction). One produces a vector bundle $\hat{E}$ on $X^*$ by solving the Dirac equation in the background of a perturbation of $\nabla$ parameterized by $X^*$. Projecting on $\hat{E}$ the trivial connection of the trivial $L^2$ bundle, one obtains the transformed connection, and the necessary endomorphisms are produced by projecting the multiplication by the coordinates of the non-compact directions of $X$.

The transform provides a framework in which to think about the classification of all the finite $L^2$-norm solutions to the ASD equation. This framework guided several authors in the understanding of moduli spaces of instantons (or their appropriate dimensional reduction) on various quotients $X$ of $\mathbb{R}^4$: [ADHM78, Hit83, CG84, Sch88, BvB89, HM89, HM90, Nak93, BJ01, Nye01, Jar02b, CK03, Cha06, Sza07, CH09]. Notice the gentle progression in time. It is not over yet: the underlined cases in the diagram above have not yet been fully investigated.

The heuristic has been implemented for other family of spaces, notably for the asymptotically locally Euclidean complete hyper-Kähler manifolds (the *ALE gravitational instantons*) by Kronheimer–Nakajima in [KN90], recently for some asymptotically locally flat hyper-Kähler manifolds by [Che09], and partially for Higgs bundles in [FJ08].

The *Fourier–Mukai transform*, algebraic counterpart and equivalent in certain circumstances to the Nahm transform, was introduced in the early 80's as a reciprocity between derived categories on Abelian varieties. Its theory (see e.g., [BBR09]) has reached a more mature stage than the theory for the Nahm transform.

# 1 Accomplishments in geometry

**Foundational result about the Nahm transform for calorons:** The joint work [CH09] with Jacques Hurtubise completes the work begun by Nye and Singer [Nye01, NS00] concerning the Nahm transform for instantons on $S^1 \times \mathbb{R}^3$, the so-called *calorons*. They considered the Nahm transform directly, and do most of the work required to show that the transform is a involutive.

The missing ingredients lie in complex geometry. In the case of a monopole $(A, \Phi)$ on $\mathbb{R}^3$, the spectral curve of the monopole is the set $\{(z,\lambda) \mid \det(\Phi(z) - \lambda \mathbf{I}) = 0\}$. Hence it represents how the spectrum of $\Phi$ gets deformed as we vary $z$. By work of Garland–Murray [GM88], we can think of a caloron as a monopole over $\mathbb{R}^3$ in a Kač–Moody algebra, and we can thus create more complicated spectral data for calorons. Using twistor theory, we proved:

**Theorem 1.1 (Charbonneau–Hurtubise [CH09])** *There is an equivalence between*

1. *Generic calorons of charge* $(k, j)$*;*
2. *Generic solutions to Nahm's equations on the circle;*
3. *Generic spectral data.*

*The equivalences of 1. and 2. are given in both directions by the Nahm transform.*

We show in [CH08b] that solutions to Nahm's equations on $S^1$, subject to the appropriate singularity behavior, are describable in terms of a geometric quotient of a family of matrices. The main tool used is a spectral sequence of Buchdahl [Buc87] giving what is called a monad description of the bundles pertinent to this story.

**Theorem 1.2 (Charbonneau–Hurtubise [CH08b])** *There is an equivalence between*

1. *Irreducible solutions to Nahm equations on the circle, with rank $k$ over $(0,\pi)$, rank $k+j$ over $(\pi, 2\pi)$, subject to some matching conditions at $\pi$ and $0 \sim 2\pi$, modulo the action of the unitary gauge group.*
2. *Vector bundles $E$ of rank two on $\mathbb{P}^1 \times \mathbb{P}^1$, with $c_1(E) = 0, c_2(E) = k$, trivialized along $\mathbb{P}^1 \times \{\infty\} \cup \{\infty\} \times \mathbb{P}^1$, and with a based flag $\phi\colon \mathcal{O}(-j) \hookrightarrow E$ of degree $j$ along $\mathbb{P}^1 \times \{0\}$ (up to non-zero scalar multiple).*
3. *Matrices $A, B$ $(k \times k)$, $C$ $(k \times 2)$, $D$ $(2 \times k)$, $A'$ $(j \times k)$, $B'$ $(1 \times k)$, $C'$ $(j \times 2)$, satisfying the monad equations*

$$[A,B] + CD = 0,$$

$$\begin{bmatrix} B' \\ 0 \end{bmatrix} A + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} A' - A'B - C'D = 0,$$

$$-\begin{bmatrix} 0 \cdots 0 & 1 \end{bmatrix} A' + \begin{bmatrix} 1 & 0 \end{bmatrix} D = 0,$$

*and some genericity conditions, modulo the obvious action of* $\mathrm{Gl}(k,\mathbb{C})$.

In summary, the Nahm transform allowed us to take a PDE with four variables to an ODE (one variable) and holomorphic techniques allowed us to reduce the question to an algebraic one.

**Corollary 1.3** *Calorons with structure group* $\mathrm{SU}(2)$ *exist for all the choice of parameters $k$ and $j$.*

To put things in perspective, the following table summarizes our knowledge about the existence of instantons of charge one on all the four-dimensional quotients of $\mathbb{R}^4$.

| $X$ | Existence of SU(2) instantons of charge one on $X$ | Result due to |
|---|---|---|
| $\mathbb{R}^4$ | Yes! (for all charge) | follows from [ADHM78] |
| $\mathbb{R}^3 \times S^1$ | Yes! (for all charge) | **Hurtubise and myself in [CH08b]** |
| $\mathbb{R}^2 \times T^2$ | sometimes | Jardim |
| $\mathbb{R} \times T^3$ | *not done yet (see Expected Result 3.2)* | **us, in progress [CHJ09]** |
| $T^4$ | never | Braam–van Baal [BvB89], Schenk [Sch88] |

The Nahm transform for $X = \mathbb{R} \times T^3$ is not yet fully proved, and thus cannot be used yet to answer existence questions. One of my ongoing project aims at proving a non-existence result in the charge one case and the project I intend to tackle afterwards is to complete the Nahm transform for $X$. Meanwhile, we still have partial information.

**Theorem 1.4 (Charbonneau [Cha06])** *Let $(E, A)$ be an $\mathrm{SU}(2)$-instanton on $\mathbb{R} \times T^3$ of asymptotic behavior determined by the set $W$ in the dual $T^3$ (there is a precise way to associate this set to a connection with flat limit). Let $(V, B, \Phi)$ be the object on the dual $T^3$ produced by the Nahm heuristic. Then*

$$\mathrm{rk} V = \frac{1}{8\pi^2} \int_{\mathbb{R}\times T^3} |F_A|^2,$$

*$(B, \Phi)$ satisfies the Bogomolny equation $d_B\Phi = *F_B$ outside of $W$. Around $w \in W$, we have projections $P_+$ and $P_-$ on subbundles such that*

$$\Phi = \frac{i}{2|z-w|}(P_+ - P_-) + O(1) \qquad \text{as } z \to w.$$

I also studied in [Cha04] the behavior of the transformed connection $B$ at $w \in W$ for particular cases. Moreover, due to our work [CH08a], we know a lot about moduli spaces of singular monopoles (solutions to Bogomolny equation) on $T^3$, and in fact on any flat $S^1$ bundle on a Riemann surface. On $S^1 \times \Sigma$, we consider monopoles with structure group $U(n)$ that admit singularities: near the singularity, the monopole should decompose into a sum of Dirac monopoles. This condition was studied by Kronheimer [Kro86] and Pauly [Pau98] who show, by exploiting the geometry of the Hopf fibration, that there are natural local lifts from $\mathbb{R}^3$ to $\mathbb{R}^4$ that tame the singularity.

The solutions to the Bogomolny equation we consider thus have Dirac-type singularities at fixed points $p_i = (t_i, z_i) \in S^1 \times \Sigma$, where the $z_i$ are distinct. Let $R_i$ be the geodesic distance, in $S^1 \times \Sigma$, to $p_i$. Near the singularity $p_i$, the Higgs field $\Phi$ is asymptotic to $\mathrm{diag}(k_{i1}, \dots, k_{in})\sqrt{-1}/2R_i$, where $\vec{k}_i = (k_{i1}, \dots, k_{in})$ is a sequence of integers. We collect the pertinent data in two sequences $\mathbf{K} = \big((\vec{k}_1, z_1), \dots, (\vec{k}_N, z_N)\big)$, and $\vec{t} = (t_1, \dots, t_N, T)$. Denote by $E_t$ the restriction to $\{t\} \times \Sigma$ of the bundle $E$ on which the monopole is defined. Let the degree of $E_0$ be $k_0$. As one moves through the point $p_i$, the degree of $E_t$ changes by $\mathrm{tr}(\vec{k}_i) := \sum_j k_{ij}$; in particular, it must be that $\sum_i \mathrm{tr}(\vec{k}_i) = 0$.

**Theorem 1.5 (Charbonneau–Hurtubise [CH08a])** *The moduli space $\mathfrak{M}_{k_0}(S^1 \times \Sigma, p_1, \dots, p_N, \vec{k}_1, \dots, \vec{k}_N)$ of $U(n)$ monopoles on $S^1 \times \Sigma$ with $E_0$ of degree $k_0$ and singularities at $p_j$ of type $\vec{k}_j$ is isomorphic to the space $\mathfrak{M}(\Sigma, k_0, \mathbf{K}, \vec{t})$ of $\vec{t}$-stable[1] holomorphic pairs $(\mathcal{E}, \rho)$ with*

- *$\mathcal{E}$ a holomorphic rank $n$ bundle of degree $k_0$ on $\Sigma$,*
- *$\rho$ a meromorphic section of $\mathrm{Aut}(\mathcal{E})$ such that* $\mathrm{divisor}(\det(\rho)) = \sum_i \mathrm{tr}(\vec{k}_i) z_i$ *and $\rho$ is conjugate to* $\mathrm{diag}_j(z^{k_{ij}})$ *near $z_i$.*

# 2 Accomplishments in interdisciplinary work

## 2.1 Statistics

At McGill, I assisted Juli Atherton in a key optimization lemma of her thesis. My main scientific contribution to this project was to adapt the typical curvature/shape relationship well known for surfaces to this multidimensional setting and prove the domain had just the right shape to allow her to use convex optimization results.

Atherton's doctoral thesis described in our joint paper [ACW+09] was awarded the *Pierre Robillard Award*. This prize is awarded each year by the Statistical Society of Canada (SSC) to the author of the best doctoral thesis in probability or statistics defended at a Canadian university.

## 2.2 Probability

At McGill, Paul Tupper and I mentored a student in a summer project. In the paper [CST09] resulting from this project, we consider the weak convergence of numerical methods for stochastic differential equations (SDEs). Weak

[1] The required notions, in particular of stability, are defined in [CH08a].

convergence is usually expressed in terms of the convergence of expected values of test functions of the trajectories. Here we present an alternative formulation of weak convergence in terms of the Prokhorov metric on spaces of random variables and establish bounds on the rates of convergence in terms of this metric for a general class of methods, showing how the bounds on the error depend on the smoothness of the test functions.

### 2.3 Statistical mechanics and hard sphere crystallization

My colleague Patrick Charbonneau (professor in the Duke Chemistry department) studies glass formation and crystallization. He recently headed an international collaborative effort to understand the crystallization of hard spheres in higher dimensions, in order to explain the extraordinary stability of the fluid phase observed in simulated systems of up to $6D$. The work challenged hypotheses used to rationalize crystal formation by studying them in higher dimension. We were particularly successful in clarifying the role of geometric frustration, i.e., the mismatch between local fluid and global crystal order. In our paper [vMCFC09], I contributed to the development of higher-dimensional intuition and tools. In particular, I simplified and generalized the standard bond-order parameters used in the physics and chemistry literature to detect if clusters of hard spheres are in fluid or crystalline state.

As a follow up on this work, I am currently mentoring a local high school student on a modest project to understand some of the combinatorics of bond-order parameters.

## 3 Current research

**2.1 Existence and non-existence of spatially periodic instantons (with J. Hurtubise and M. Jardim).**

For SU(2)-instantons of charge one on $\mathbb{R}\times T^3$, it is natural to expect a result half-way between the absolute obstruction to their existence on $T^4$ ([BvB89]) and the mixed existence/non-existence result ([Jar02a]) for $\mathbb{R}^2\times T^2$.

After a gauge transformation, an instanton $A$ on $\mathbb{R}\times T^3$ can be seen as a flow of connections on $T^3$. In this gauge, a finite charge instanton then limits to flat SU(2)-connection on $\{\pm\infty\}\times T^3$. Up to gauge equivalence, flat $U(1)$-connections on $T^3=\mathbb{R}^3/\Lambda$ are parameterized by $\mathbb{R}^3/\Lambda^*$: given $w\in\mathbb{R}^3$, we define the trivial $\mathbb{C}$-bundle $L_w$ with connection $\omega_w = 2\pi i\sum w_j dx^j$; different representatives of $w\in\hat{T}^3=\mathbb{R}^3/\Lambda^*$ give equivalent bundles and connections. For some $w_\pm\in\hat{T}^3$, the instanton $A$ on $\mathbb{R}\times T^3$ limits to $\omega_{w_\pm}\oplus\omega_{-w_\pm}$ at $\pm\infty$, or equivalently, $E|_{\{\pm\infty\}\times T^3}=L_{w_\pm}\oplus L_{-w_\pm}$.

Let's think of $\mathbb{R}\times T^3$ inside $\mathbb{CP}^1\times T^2$. The inclusion does not preserve the ASD condition, but it is holomorphic. In preliminary work, we have produced, a holomorphic sheaf $\mathcal{E}$ on $\mathbb{CP}^1\times T^2$ extending $(E,\bar\partial_A)$.

**Expected Result 3.1** *Let $(E,A)$ be a rank 2 instanton of charge $k$ on $\mathbb{R}\times T^3$, with asymptotic behavior given by $w_\pm=(\alpha_\pm,\xi_\pm)\in S^1\times T^2$. Then there is a holomorphic bundle $\mathcal{E}$ on $\mathbb{CP}^1\times T^2$ such that*

$$\mathcal{E}|_{\mathbb{R}\times T^3}\cong(E,\bar\partial_A),\qquad \mathcal{E}|_{\{\pm\infty\}\times T^2}\cong L_{\xi_\pm}\oplus L_{-\xi_\pm}$$

*and such that $c_1(\mathcal{E})=0$ and $c_2(\mathcal{E})=k$, and $\mathcal{E}$ is stable, for some appropriate notion of stability.*

The idea is to shift the problem to the holomorphic category and study stable holomorphic bundles. The difficulties occur because $\mathbb{R}\times T^3$ is not compact. It that sense, this result fits well with the various results obtained in the literature for instantons on cylindrical manifolds and on manifolds with cylindrical ends in [Guo96, Kov95, KM93, Owe01, Tau93] and is inspired by the similar result of [BJ01] for $\mathbb{R}^2\times T^2$.

In this holomorphic category, it is easier to detect obstructions to the existence of certain solutions using techniques of Braam–Hurtubise [BH89]: we look at $\mathcal{E}$ on $\mathbb{CP}^1\times T^2$ as a $\mathbb{CP}^1$ family of rank 2 bundles on $T^2$; this family can be described by a map $\mu\colon\mathbb{CP}^1\to\mathbb{CP}^1$ and the degree of this map is roughly equal to the charge of the instanton.

**Expected Result 3.2** *If $w_\pm=(\alpha_\pm,\xi_\pm)$ satisfies either $\{\pm\xi_+\}=\{\pm\xi_-\}$ then the degree of the map $\mu$ must be greater of equal to two. Hence there are no charge one rank two instanton on $\mathbb{R}\times T^3$ with limiting behavior $w_\pm$.*

**2.2 Moduli space of $G_2$ and Spin(7) Donaldson–Thomas connections (with S. Karigiannis).** Despite their relevance in string theory (see [AOS97, BKS98, Pop09]), we know relatively little about $G_2$ and Spin(7)

Donaldson–Thomas instantons. The field is however rapidly evolving. An important part of the analytic framework was worked by Tao and Tian in [TT04, Tia00]. A non-trivial existence result is found in the thesis of Lewis [Lew99] on a specific compact eight-manifold of holonomy Spin(7). These past months, the papers [HILP09, Pop09] have studied $G_2$ and Spin(7) DT-instantons that are invariant under some group of isometry of the underlying manifolds. There are no genuinely 7-dimensional example of irreducible $G_2$ DT-instantons known yet, but important partial work was completed very recently by Sá Earp in his thesis [SE09].

A similar story is simultaneously developing for manifolds with holonomy $\mathrm{SU}(n)$ and the appropriate generalization of the notion of Donaldson–Thomas connections with the recent work of Tanaka [Tan, Tan08a, Tan08b] and

A general guide of what has to be done and what are the hopes of the theory was recently proposed by Donaldson and Segal in [DS09]. There is in particular a duality between DT-instantons and calibrated sub-manifolds.

To highlight an important difference with the four-dimensional instanton story, it is important to note that we cannot start our investigation with DT-instantons on $\mathbb{R}^8$ since, as Jaffe–Taubes pointed out in [JT80], there are no nonflat critical points of the Yang–Mills functional on $\mathbb{R}^8$ with finite $L^2$-norm. We can however find examples of Donaldson–Thomas instantons on $T^8$ so it is a natural choice to work on this space. The story should parallel in some way the $T^4$ story. We therefore propose to study this moduli space using a modified version of the Nahm transform that has been so fruitful in the case of $T^4$.

# 4 Future research

**3.1 Nahm transform on $\mathbb{R} \times T^3$ and $T^3$ (with J. Hurtubise).** Apart from some numerical approximations and remarks in [vB96] and a computation of the Nahm transform of charge 1 instantons in [vB99], no progress has been made in the case of $\mathbb{R} \times T^3$ prior to my work [Cha04, Cha06]. The space $\mathbb{R} \times T^3$ is in relation via the Nahm transform with $T^3$. Since $T^3$ is three-dimensional, the relation equation is the Bogomolny equation $*F_B = d_B\Phi$ and its solutions $(B, \Phi)$ are called *monopoles*. The endomorphism $\Phi$ of the bundle is called the *Higgs field*.

In an effort to simplify the notations, let's denote $\mathfrak{M}$ the various moduli spaces

$$\begin{aligned}
\mathfrak{M}(\mathbb{R} \times T^3) &= \text{ moduli space of instantons,}\\
\mathfrak{M}(\mathbb{P}^1 \times T^2) &= \text{ moduli space of parabolic bundles,}\\
\mathfrak{M}(\hat{T}^3) &= \text{ moduli space of singular monopoles,}\\
\mathfrak{M}(\hat{T}^2) &= \text{ moduli space of polystable Higgs pairs.}
\end{aligned}$$

We can use labels suggestive of the phrases "Nahm transform," "Higgs pairs," "Spectral curve" and "Compactification" for maps between these moduli spaces. These maps and moduli spaces fit in the diagram

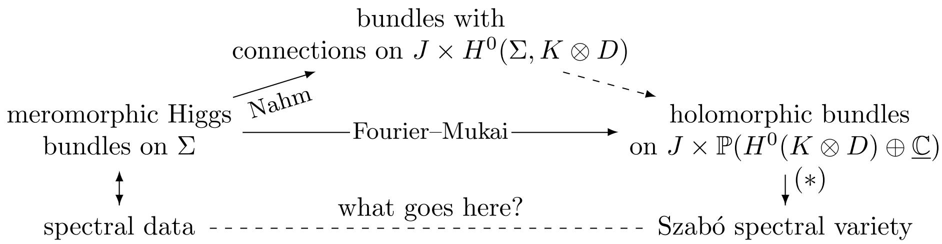
$$\begin{array}{ccc}
\mathfrak{M}(\mathbb{R} \times T^3) & \xrightarrow{\mathcal{N}} & \mathfrak{M}(\hat{T}^3)\\
\mathcal{C}\big\downarrow & & \big\downarrow\mathcal{H}\\
\mathfrak{M}(\mathbb{P}^1 \times T^2) & \xrightarrow[\mathcal{S}]{} & \mathfrak{M}(\hat{T}^2).
\end{array} \tag{1}$$

We do have to add a bit of decoration (location of singularities, Chern class, etc.) on all those $\mathfrak{M}$ for these statements to make sense but for the purpose of this statement, this schematic is sufficient.

The map $\mathcal{N}$ was studied in [Cha06]. The map $\mathcal{C}$ we are currently studying in [CHJ09] and it appears to be an isomorphism. The map $\mathcal{H}$ was studied in [CH08a] and was shown to be an isomorphism. The map $\mathcal{S}$ is what is left to be defined. We have a candidate and we can see that the Diagram (1) is commutative. Proving that $\mathcal{S}$ is an isomorphism, we will be able to prove that the Nahm transform $\mathcal{N}$ is an isomorphism.

**3.2 Fourier–Mukai/Nahm transform of meromorphic Higgs bundles (with M. Jardim and R. Moraru).** Let $\mathcal{E} \to \Sigma$ be a Hermitian vector bundle equipped with a unitary connection $\nabla$ and a bundle map $\theta \colon \mathcal{E} \to \mathcal{E} \otimes K_\Sigma$ satisfying the dimensional reduction of the ASD equation to dimension two (the Hitchin's equations). The triple $(\mathcal{E}, \nabla, \theta)$ is called a *Higgs bundle* and the moduli space of them were studied extensively in the seminal paper [Hit87] of Hitchin. They remain to this day an intensive research topic.

The goal of this project is to investigate the following diagram.




For most of these maps, not much is known. Some informal notes [Sza] describe in details the arrow $(*)$. In the case of holomorphic Higgs bundle, the Fourier–Mukai map was described in 2006 work [Bon06] of Bonsdorff and the Nahm transform map was described in 2008 by [FJ08]. We do not understand yet how to build the inverse transform. In the meromorphic case, the Nahm transform was established in the case $g = 0$ by Szábo in [Sza07] and in the case $g = 1$ by Jardim in [Jar99]. In those cases, contrary to the general genus holomorphic case, we actually understand very well the inverse transform.

**3.3 Other long term projects** In this section, I describe very quickly extra problems I have in the back of my mind but have not had time to explore enough.

- The moduli space of calorons must theoretically support a hyperkähler structure and most probably an integrable system. These aspects are not explicitly addressed in our work [CH08b, CH09]. I have started to explore these aspects with J. Hurtubise and R. Moraru.
- The Nahm transform was extended to hyperkähler manifolds (gravitational instantons) of type ALE (asymptotically locally Euclidean) by Kronheimer–Nakajima in [KN90] and Cherkis proposed an extension for those of type ALF in [Che09]. To be able to extend more largely the domain of definition of this tool, the first step is finish the proof of the heuristics in the base case of quotients of $\mathbb{R}^4$. The case of monopole sheets on $\mathbb{R} \times T^2$, to my knowledge largely ignored up to known because of technical difficulties, is an ideal thesis problem.
- The question of finding the Poincaré polynomial of the moduli space of calorons is relevant to physics. Using the description given in [CH09], and ideas of [Hau06, Hau08], we should be able to perform this "caloron counting." There are however difficulties due to the format of our equations.
- In the background is lurking the fundamental question of the classification of the gravitational instantons and of Yang–Mills intantons on those spaces. Good progress was done in the past four years by Cherkis, Etesi, Jardim, and Szabó in [Che09, CH05, EJ08, ES08].

# 5 Plans for the training of students

Here are a few sample projects I have in mind for students.

1. (Undergraduate Research project) Experiment with MAPLE using a lattice approximation to study the solutions to the Bogomolny equations on $T^3$ numerically and getting pictures of the energy $|\Phi|$ of the Higgs field of the monopole.
2. (Undergraduate Research project) Implement in MAPLE, or a similar symbolic computing software, the inverse Nahm transform from $S^1$ to $S^1 \times \mathbb{R}^3$. More precisely, given matrices satisfying the conditions of Theorem 1.2, produce the instanton on $S^1 \times \mathbb{R}^3$. To represent that instanton visually, create pictures of the action density $S := |F_A|^2$.
3. (M.Sc. Thesis project) Rewrite Hurtubise and Murray's construction of SU$(n)$-monopoles without the use of the spectral curves, following Nakajima's example [Nak93].
4. (M.Sc. Thesis project) Consider the decay of the curvature $F_A$ for an instanton on the various four-dimensional quotients of $\mathbb{R}^4$.

5. (Ph.D. Thesis project) Develop the proper setup for the Nahm transform on the last quotient of $\mathbb{R}^4$ not treated, $\mathbb{R} \times T^2$, and prove it is a bijection.

6. (Ph.D. Thesis project) Study moduli spaces of instantons on ALF gravitational instantons.

7. (Ph.D. Thesis project) Use moduli spaces of singular instantons on $\mathbb{R}^2 \times T^2$ to produce examples of ALG gravitational instantons.

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