Fredholm theory on quasi-asymptotically conical manifolds

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(joint work with Rafe Mazzeo)

Recall that an operator between Hilbert spaces is called Fredholm if it has finite dimensional kernel, closed range, and finite dimensional cokernel. To it one associates the index, which is the dimension of the kernel minus the dimension of the cokernel.

On a compact Riemannian manifold (X, g), an elliptic operator is always Fredholm as an operator acting between the L^2-spaces. Moreover, the index is a topological object, and it is given by the Atiyah-Singer index theorem. For example, the Dirac operator

$$\not{D}_E^+ : H^1(X, S^+ \otimes E) \to L^2(X, S^- \otimes E) \tag{1}$$

twisted by some Hermitian bundle E is Fredholm, and its index is

$$\operatorname{index} \not{D}_E^+ = \int_X \hat{A}(g)\mathrm{ch}(E). \tag{2}$$

When we have a non-compact manifold, ellipticity is not enough to guarantee Fredholmness, as usually the range of the operator is not closed. However, choosing the right weights for the Sobolev spaces, usually allows to conclude that the operator is Fredholm. For example, in the case when (X, g) is an asymptotically conical manifold – a non-compact manifold with the infinite end being asymptotically a cone – one introduces a weight which has to do with the distance ρ on the infinite end of X, and then the Laplace operator (twisted by some bundle) is Fredholm as an operator

$$\Delta_E : \rho^\delta H^2(X, E) \to \rho^{\delta-2} L^2(X, E) \tag{3}$$

if and only if δ is not an indicial root of Δ_E. For the Dirac operator one has

$$\not{D}_E^+ : \rho^\delta H^1(X, S^+ \otimes E) \to \rho^{\delta-1} L^2(X, S^- \otimes E), \tag{4}$$

and this is Fredholm if and only if again δ is not an indicial root of $\not{D}_E^+$. The Atiyah-Patodi-Singer index theorem gives the index of this operator, as the integral contribution of the compact version, plus a contribution of the boundary at infinity, contribution which is given by the eta-invariant,

$$\operatorname{index} \not{D}_E^+ = \int_X \hat{A}(g)\mathrm{ch}(E) - \frac{\eta_E}{2}. \tag{5}$$

In this talk we are concerned with a new class of non-compact manifolds which we call "quasi-asymptotically conical", or QAC for short. Our ultimate goal is to generalize the formulas (2) and (5) to the class of these manifolds. For now, we are concerned with the more modest goal of figuring out the spaces of functions for which the Laplace operator (and other geometrical operators) are Fredholm, thus generalizing (3) to this new class of manifolds.

Before diving into the technical definition of the QAC spaces, let us first present a bit of motivation, and show why these spaces deserve to be looked at.

The QAC spaces arise naturally as resolutions of singularities in algebraic geometry. Locally, a complex orbifold is modeled on $\mathbb{C}^n/G$, with G a finite subgroup of $U(n)$. Note that the origin of $\mathbb{C}^n$ always gives a singular point in $\mathbb{C}^n/G$. Depending on the way G acts on $\mathbb{C}^n$, we might have some other singular points or not. A resolution of singularities of $\mathbb{C}^n/G$ is a pair (X,π), with X a smooth complex manifold of dimension n, and $\pi : X \to \mathbb{C}^n/G$ a proper surjective map that is a biholomorphism between dense open sets. If the origin gives the only singular point of $\mathbb{C}^n/G$, then X is a non-compact manifold whose geometry is $(\mathbb{C}^n \setminus B_R(0))/G$ outside a compact set. Such a geometry is an example of asymptotically conical manifold. On the other hand, the action of G on $\mathbb{C}^n$ might have more singular points, and then the singular set is non-compact (it arises from subspaces of $\mathbb{C}^n$ with non-trivial stabilizers under the action of G). By resolving the singularities of such manifolds, one is lead to consider the notion of "quasi-asymptotically conical manifolds", geometries which outside a compact set are composed of pieces which are either cones over (possibly) singular spaces, or products between such cones and euclidean spaces.

We introduce three types of spaces which are closely related to each other: (1) the class $\mathcal{I}$ of *iterated cone-edge spaces*, singular spaces obtained via an iterated coning procedure; (2) the class $\mathcal{D}$ of *resolution blowup spaces*, a class of smooth spaces which arise as smoothings of spaces in $\mathcal{I}$; and (3) the class $\mathcal{Q}$ of *quasi-asymptotically conical spaces*, noncompact spaces which on the infinite end have as link an element in $\mathcal{D}$. Basically, if $(Y_0, h_0) \in \mathcal{I}$ is an iterated-cone edge singular space, then a smooth compact manifold Y is in $\mathcal{D}$, if there exists a family of metrics $\{h_\epsilon\}$ on Y so that $(Y, h_\epsilon) \to (Y_0, h_0)$ in Gromov-Hausdorff sense. We call (Y, h_ϵ) a resolution blowup space associated to (Y_0, h_0). Such a space comes with a radius function w_ϵ which converges to s, the distance to the singular stratum of (Y_0, h_0), as $\epsilon \to 0$. On the other hand, a QAC space is a smooth manifold (X, g) with the metric outside a compact set asymptotic to

$$d\rho^2 + \rho^2 h_{1/\rho},$$

meaning that the link at radius ρ is the resolution blowup space $(Y, h_{1/\rho})$. It comes with a pair of two radius functions (ρ, w), with $\rho : X \to [1, +\infty)$ the distance on the quasi-asymptotically conical end, and with w at radius ρ being the radius function $w_{1/\rho}$ on the resolution blowup space $(Y, h_{1/\rho})$.

Note that the construction of resolution blowup and QAC spaces is an inductive one. Once we constructed a QAC space, we can go on and construct a resolution blowup space for an iterated cone-edge space with higher depth singularities – thus a *depth* induction. As such, to prove a Fredholmness result on a QAC space (X, g), one first need to show that the restriction of the operators on each slice behaves well in the limit, meaning one needs to prove a spectral convergence result for the resolution blowup spaces (Y, h_ϵ) which come with the QAC package. This spectral result is based on the Fredholmness of the corresponding operator on the lower depth QAC spaces used to construct (Y, h_ϵ), and this is the inductive step for its proof.

Theorem 1 (Spectral Convergence)**.** *Let $(Y, h_\epsilon) \in \mathcal{D}$ be a resolution blowup space. Let L_ϵ be a generalized Laplace operator acting on the sections of a geometric vector bundle. Assume that each model operator on the lower depth QAC spaces appearing in the deconstruction of Y is positive, and that 0 is not a L^2-eigenvalue. Then the spectrum of L_ϵ converges to the spectrum of the Friedrichs extension of the limiting operator L_0 on (Y_0, h_0).*

The analogue of (3) in the QAC context is the following:

Theorem 2. *Let (X, g) be a QAC manifold with radius functions (ρ, w). Let $\mathcal{L}$ be a generalized Laplace operator on X twisted by some geometrical vector bundle E. Then*

$$\mathcal{L} : \rho^\delta w^\tau H^2(X, E) \to \rho^{\delta-2} w^{\tau-2} L^2(X, E) \tag{6}$$

is a Fredholm operator provided δ is not an indicial root for $\mathcal{L}$, and τ is so that the lower depth operators on the corresponding QAC spaces are positive and do not have 0 as a L^2-eigenvalue.

The proofs of these two theorems are interlinked, and they go inductively. The first inductive step in the proof of Theorem 1 – the case of a resolution blowup space corresponding to a space (Y_0, h_0) with isolated conical singularities – was proved in the PhD thesis of Rowlett [6]. Then the general case (assuming Theorem 2) was presented in [5]. Note that in the context of the scalar Laplacian acting on QALE manifolds (a special class of QAC manifolds), the proof of Theorem 2 appears in Joyce [4]. Since it is based on the maximum principle, his proof cannot be generalized to the case of systems. In the process of proving Theorem 2 we also prove a similar result for weighted Hölder spaces.

References

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