A Two-Time-Level Semi-Lagrangian Double Fourier Method—A Formulation

Anita T. Layton

Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina, U.S.A. (layton@amath.unc.edu).

1 Introduction

The accuracy of global weather and climate models depends on many factors, including the accuracy of the knowledge of the state of the atmosphere at the initial time, the numerical methods employed, and the resolution. Climate and weather prediction computations are known to be very time-consuming. In particular, a long-standing problem in the integration of numerical weather prediction models is that with explicit Eulerian time discretization methods the maximum permissible time step is restricted by stability rather than accuracy. That is, in order for the integration to be stable, the time step has to be so small that the time truncation error is much smaller than the spatial truncation error, resulting in high computation cost. Early models used an explicit leapfrog method, in which the time step is limited by both the Courant-Friedrichs-Lewy (CFL) condition as well as the propagation of gravity waves. Discretization schemes based on the semi-Lagrangian treatment of advection offer the promise of larger time steps, with no loss in accuracy compared to Eulerian-based advection schemes [11, 14]. Since gravity terms may render the equations stiff and thus severely restrict the time step even with semi-Lagrangian advection approximations, one needs to combine the semi-Lagrangian formulation with semi-implicit time-stepping to obtain maximum benefit from the semi-Lagrangian approach [9, 12]. By combining a semi-Lagrangian treatment of advection and a semi-implicit treatment of gravity terms, it is possible to increase the time step substantially while maintaining numerical stability [10, 11].

In most global atmospheric applications, spatial discretization schemes are based on the spectral transform method [5, 15], in which solution fields are expressed as spherical harmonic expansions. Since the spherical harmonics are the natural representation of a two-dimensional field on the surface of a sphere, the spectral approach provides an elegant solution to the *pole problem*, including the fact that some variables (e.g., the longitudinal and latitudinal velocity components) may be multi-valued at the poles. Also, since the spherical harmonics are eigenfunctions of the spherical Laplace operator, the resulting semi-implicit Helmholtz problem is trivial to solve. Another advantage of the spectral transform method is that, provided the solution is sufficiently smooth, the method generates numerical approximations with exponential convergence and thus with accuracy higher than most other methods (e.g., finite difference methods) at the same spatial resolution.

Although the spectral transform method seems ideal for the spherical domain, it is computationally expensive, especially at high spatial resolutions, since it requires associated Legendre transforms. In the case of Fourier transforms in the longitudinal direction, fast Fourier Transforms (FFTs) may be used and their computational cost grows as $\mathcal{O}(N^2 \log N)$, where N is the number of subintervals in one spatial dimension. Efficient associated Legendre transforms, analogous to FFTs, have not yet been developed for resolutions of current interest. Therefore, the associated Legendre transforms are often performed by summation and their cost is $\mathcal{O}(N^3)$. Thus there is interest in the atmospheric community in developing alternative numerical methods that have stability and accuracy comparable to that of the spectral transform method but have lower computational cost.

Numerical methods based on Fourier series, rather than spherical harmonics, have been proposed as a viable alternative to the spectral transform method, in both pseudospectral [7, 13] and spectral [1, 2, 8, 16] forms. Yee solved the Poisson equation by means of a method based on double Fourier series on a spherical surface [16], using sine and cosine series as latitudinal basis functions for odd and even zonal wave numbers, respectively. Cheong applied a modified double Fourier series method to generate solutions of the elliptic and vorticity equations [2] and the shallow water equations (SWEs) [1]. His method is similar to that of Yee [16], but with somewhat different basis functions for even zonal wavenumbers. The advantage of using the double Fourier series is that FFTs can be used in both the longitudinal and latitudinal directions, rather than the associated Legendre transforms used by the spectral transform method in the latitudinal direction, thus resulting in a significant reduction in computational cost. As with the spectral transform method, inversion of the Laplace operator is $\mathcal{O}(N^2)$ for the double Fourier method, making it an attractive candidate for semi-implicit formulations and the resulting Helmholtz equation.

Disadvantages of the double Fourier series are that it permits discontinuities at the poles and the nonlinear terms in the equations give rise to non-isotropic waves which may lead to numerical instability. Both problems can be remedied by applying a spherical harmonic projection [13] (i.e., by performing a least-squares projection of the prognostic variables onto the spherical harmonics) or a diffusive filter [1]. The use of a spherical harmonic projection allows one to obtain solutions that have the same accuracy of those obtained by means of the spherical harmonic spectral transform method, but has the disadvantage of re-introducing associated Legendre transforms into the dynamics and thus increasing the computational cost.

In [6], a three-time-level semi-Lagrangian semi-implicit (SLSI) double Fourier method was presented. The Lagrangian nature of the method maintains numerical stability while allowing time steps up to ∼30 times larger than the maximum time step permissible in an Eulerian-based method. Provided that the solution is sufficiently smooth, the method generates approximations with third-order accuracy. Because the double Fourier expansion permits discontinuities at the poles and nonisotropic waves, the spherical harmonic projection [13] is used; that is, the prognostic variables are projected onto the spherical harmonic space at the end of every time step. Aliasing is controlled by means of a quadratic truncation grid. The associated Legendre transforms required in the spherical harmonic projections unfortunately increases the computational complexity of the method from $\mathcal{O}(N^2 \log N)$ to $\mathcal{O}(N^3)$. Nonetheless, a total of six associated Legendre transforms are required for each time step, which still offers a speedup of 25% compared to the eight associated Legendre transforms required for the most efficient implementation of the semi-Lagrangian spectral transform method.

Generally, a two-level semi-Lagrangian semi-implicit method is preferred to a three-level method. Indeed in [3] Cote and Staniforth developed a method based on the two-time-level SLSI method and the spherical harmonic spectral transform method, and showed that their method is twice as efficient as a similar method based on the three-time-level SLSI method. Thus below we discuss the possible formulations of a two-time-level SLSI double Fourier series for obtaining solutions of the SWEs in spherical coordinates.

2 Model Equations

Because the earth is approximately spherical, most global atmospheric models in use today are based on spherical coordinates. To define the SWEs on the sphere, let $0 \leq \lambda < 2\pi$ be longitude and $0 \leq \theta \leq \pi$ be colatitude; let $\vec{v}$ denote the vector (u, v), where u and v are the wind velocity components in the longitudinal and colatitudinal directions, respectively; ϕ' be the geopotential perturbation from the mean geopotential ϕ^*, which is assumed to be constant; a be the radius of the earth; Ω be its rotational speed; and $f \equiv 2\Omega \sin\theta$ be the Coriolis parameter. Since u and v are multi-valued at the poles, we adopt the approach of Côté and Staniforth [4] and compute the components of the wind images instead: $U \equiv u \sin\theta$ and $V \equiv v \sin\theta$. Using this notation, the vector form of the SWEs in spherical coordinates is given by

$$\sin\theta \frac{\partial \vec{v}}{\partial t} + f \sin\theta \hat{k} \times \vec{v} + \sin\theta \vec{\nabla}\phi = 0, \tag{1}$$

$$\frac{\partial \phi'}{\partial t} + (\phi' + \phi^*)\vec{\nabla} D = 0, \tag{2}$$

where $\hat{k}$ is the outward radial unit vector, the divergence $D \equiv \vec{\nabla} \cdot \vec{v}$, and

$$\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial t} + (\vec{v} \cdot \vec{\nabla}), \quad \vec{\nabla} \equiv \frac{\hat{\imath}}{a\cos\theta}\frac{\partial}{\partial\lambda} + \frac{\hat{\jmath}}{a}\frac{\partial}{\partial\theta}. \tag{3}$$

Note that (1) differs from the standard form in that it is scaled by $\sin\theta$. This prevents $\sin\theta$ from appearing in the denominator of any term and facilitates the use of standard trigonometric identities in the double Fourier spectral method.

3 Formulations for Two-Time-Level Scheme

In a three-time-level scheme, the Coriolis terms in the motion equations (1) are evaluated explicitly at trajectory midpoints at t_n; in a two-time-level scheme, the Coriolis terms may be averaged along fluid trajectories. Côté and Staniforth [3] describe such a two time-level semi-Lagrangian semi-implicit scheme for spherical harmonic spectral models. Letting $\zeta_m = [\zeta_{m,m} \;\; \zeta_{m,m+1} \cdots \zeta_{m,N-1}]^{\mathrm{T}}$, etc., they developed a linear system for each zonal wave number m,

$$
\begin{aligned}
\mathbf{A}_m\zeta_m + \mathbf{B}_m\delta_m &= \mathbf{L}_m, && (4)\\
-\mathbf{B}_m\zeta_m + \mathbf{A}_m\delta_m + \mathbf{C}_m\phi'_m &= \mathbf{M}_m, && (5)\\
\tilde{a}\delta_m + \phi'_m &= \mathbf{Q}_m. && (6)
\end{aligned}
$$

For a spherical harmonic expansion, $\mathbf{A}_m$ and $\mathbf{C}_m$ are diagonal and $\mathbf{B}_m$ is tridiagonal. $\mathbf{A}_m$ is, in fact, the discrete Laplace operator for zonal wave number m. For the double Fourier expansion, $\mathbf{A}_m$ and $\mathbf{C}_m$ are tridiagonal.

The vorticity and geopotential can be eliminated from (4)–(6) to yield the system

$$
\begin{aligned}
\zeta_m &= \mathbf{A}_m^{-1}\left(\mathbf{L}_m - \mathbf{B}_m\delta_m\right), && (7)\\
\phi'_m &= \mathbf{Q}_m - \tilde{a}\delta_m, && (8)\\
[\mathbf{A}_m - \tilde{a}\mathbf{C}_m + \mathbf{B}_m\mathbf{A}_m^{-1}\mathbf{B}_m]\delta_m &= \mathbf{M}_m - \mathbf{C}_m\mathbf{Q}_m + \mathbf{B}_m\mathbf{A}_m^{-1}\mathbf{L}_m. && (9)
\end{aligned}
$$

In theory, equation (9) can be solved for δ_m, which can then be used to form the right hand sides of equations (7)–(8). When spherical harmonics are used, the matrix on the left hand side of (9) is trivial to compute because $\mathbf{A}_m$ is diagonal. However, for the double Fourier expansion, the tridiagonal $\mathbf{A}_m$ becomes a full matrix when it is inverted, resulting in a full $(N-m)\times(N-m)$ matrix on the left hand side of (9). Thus with a double Fourier expansion, solving (7)–(9) for all m becomes an $\mathcal{O}(N^3)$ algorithm.

Alternatively, an iterative solver may be used and the resulting computations may be $\mathcal{O}(N^2)$, provided that the number of iterations required for convergence is independent of the spatial resolution. Nonetheless, the double Fourier equivalent of [3] appears to present a number of computational challenges. An alternative to [3] is to rewrite the the Coriolis terms in (1) as Lagrangian derivatives. This approach may lead to an uncoupling of the motion equations and a potentially more efficient implementation of the two-time-level SLSI double Fourier method.

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