

Homework 6: Conjugate Gradient

Due: *Thursday, October 9, 2008*

1. Read Health 11.5.5, and the article by J. R. Shewchuk on Conjugate Gradient.
2. Convergence property of a τ -random matrix. Let A be an 500×500 symmetric matrix structured as follows:
 - First we put 1 at each diagonal position and a random number from the uniform distribution on $[-1,1]$ at each off-diagonal position, while maintaining the symmetry $A^T = A$.
 - Then we replace each off-diagonal entry with $|a_{i,j}| > \tau$ by zero, where τ is a parameter.

For small τ , A is a well-conditioned positive definite matrix whose density of nonzero entries is approximately τ . As τ increases, both the condition number and the sparsity deteriorate.

Implement Conjugate Gradient to compute $A\vec{x} = \vec{b}$, where the entries of b are all ones.

Determine the value of τ_0 at which CG fails to converge. Why does CG fail at $\tau > \tau_0$? Do this by solving the equation starting with $\tau = 0.01$ and incrementing by 0.01. Count the number of floating-point multiplications. Record the residual ($|r_k|_2$) at each iteration. Plot on the same graph $|r_k|_2$ versus k for $\tau = 0.01, 0.05, 0.1, 0.2$.

Turn in print-outs of your code and the graph.

Assuming that the Cholesky factorization requires $n^3/6$ flops, how does the operation count of CG compare at $\tau = 0.01$?

3. Preconditioning. Let A be a 1000×1000 symmetric matrix whose entries are all zero except for

$$a_{i,i} = 0.5 + \sqrt{i}, \quad a_{i,j} = 1, \quad |i - j| = 1, \quad a_{i,j} = 1, \quad |i - j| = 100$$

Consider the solution of $A\vec{x} = \vec{b}$, where $b = (1, 1, \dots, 1)^T$.

- (a) Solve $A\vec{x} = \vec{b}$ using Conjugate Gradient. Record $|r_k|$ versus k .
- (b) Solve $A\vec{x} = \vec{b}$ using Preconditioned Conjugate Gradient, taking $M = \text{diag}(A)$. Plot $|r_k|$ versus k on the same graph as (a).