Math 224: Scientific Computing

Homework 11: Newton's Method, II

Due: *Thursday, November 20, 2008*

1. Read Health 5.6.1–5.6.4
2. Download a copy of `complex.hpp` from `http://www.math.duke.edu/~alayton/math224/complex.hpp`.
3. Modify one of your real-valued `newton()` routines (either `newton2()` [with the optimal value for dx] or `newton3()`) to apply to complex-valued function

```
complex cnewton(complex (*f)(complex z), complex z0);
```

4. Newton's method and fractals. As described in class, Newton's method will only converge if the initial conditions z_0 is "close enough" to the fixed point z^*. The set of all points $\{z_0\}$ that converges to z^* is called the *basin of attraction* of z^*. Consider the equation

$$f(z) = z^3 + 1 = 0.$$

Find the basin of attraction of the root $z^* = -1$ for Newton's method.

Following this outline:

   (a) Use `cnewton()` to generate a finite sequence z_k starting from an initial value z_0. Set a condition to determine if the sequence to z^* or diverging.

   (b) Repeat Step (a) starting from each point $z_0 = x_0 + iy_0$ in the domain $\{-1 \le x_0 \le 0.5, 0 \le y_0 \le 1.5\}$. Use $\Delta x = \Delta y = 0.005$ or smaller.

   (c) For each run from (b) that converges, output x_0 and y_0 to a file. (Hint: You should find that $z_0 = -0.462011 + i0.522831$ does not converge, but $z_0 = -0.222418 + i0.65408$ does.)

   Plot the set of points (the basin of attraction).

   Note that the boundary of the basin of attraction is a self-similar *fractal*. If you "zoom-in" on any part of it, you will see that it contains an infinite sequence of reduced copies of itself!

5. The return of the Wilkinson polynomial. Reconsider the Wilkinson polynomial for complex variables,

$$W_\alpha(x) \equiv (x-1)(x-2)(x-3)\cdots(x-19)(x-20) - \alpha$$

You will use `cnewton()` to obtain all the roots as a function of the parameter α, $z^{m,*}(\alpha)$ for $m = 1, 2, \ldots, 20$.

   (a) Revise your function `double w(double x);` to complex-variable form.

   (b) Starting from $\alpha = 10^{10}$, try an initial guess of z_0 to converge to the first solution $z^{1,*}(10^{10}) \approx 1$ using `cnewton()`.

   (c) Next, increase α, `alpha*=1.001` (or smaller), (up to $\alpha \le 10^{15}$) and try to find the solution at this value of α. This time, as the initial guess z_0, use the solution z^* from the previous value of α in (b). This is called a parameter-continuation method. The idea is that $z^{1,*}(\alpha_{\text{prev}})$ should be a better of the new solution than just $z_0 = 1$.

(d) But this won't help to get us the complex solutions, will it? Isn't it true that since the polynomial is purely real, and all the solutions to the some points are purely real, we are "stuck" with getting only purely-real iterates as z_k from Newton's method?

Yes, actually that's true, but easy to fix. Here's how: instead of using $z_0 = 1$ as the initial guess, or even $z_0 = z^{1,*}(\alpha_{\text{prev}})$, $\boxed{\text{try } z_0 = z^{1,*}(\alpha_{\text{prev}}) \pm i10^{-6}}$. I.e., add a small imaginary perturbation— if the answer really should be real, then this perturbation will decrease under iteration, but if the answer is complex, then this gives us just the right 'push' to get un-stuck and move to complex-valued solutions.

(e) $\boxed{\text{Plot } \mathrm{Re}(z^{1,*}(\alpha)) \text{ and } \mathrm{Re}(z^{1,*}(\alpha)) + \mathrm{Im}(z^{1,*}(\alpha)) \text{ on the same plot, as functions of } \alpha \text{ (log-}\alpha\text{ x-axis).}}$

(f) $\boxed{\text{Repeat steps (b–e) for all twenty roots.}}$ Your answer should include only ONE plot with all twenty roots plotted together. (If this is too messy, split it into a small number of plots.)

(g) One root should be approximately $z^{8,*}(8.00037\times10^{12}) \approx 8.353747443625275 - i1.056508655959121$.
$\boxed{\text{Find the value } z^{7,*}(10^{15}) \text{ (the root starting from } z^{7,*}(0) = 7\text{) as accurately as possible.}}$