Math 224: Scientific Computing

Homework 10: Newton's Method

Due: *Thursday, November 13, 2008*

1. Read Health 5.5.3–5.5.4

2. Practical convergence. We define the criterion for convergence of a sequence as

$$|x_{k+1} - x^*| = c|x_k - x^*|^\alpha, \quad 0 \le c < 1, \quad \alpha > 0$$

This is a very well motivated and clearly defined theoretical definition for the convergence of a sequence, but it is not very practical if we don't know the value x^*. Instead, let's consider a similar condition in terms of the Cauchy sequence (meaning the difference between successive numbers in the sequence),

$$|x_{k+1} - x_k| = \tilde{c}|x_k - x_{k-1}|^{\tilde{\alpha}}$$

Then the constants $\tilde{c}$, $\tilde{\alpha}$ for large k can be thought of as approximations of the real c (the asymptotic convergence coefficient) and α (the asymptotic convergence rate) without having to find x^*:

(a) Take four successive iterates from a sequence, $(x_{k-2}, x_{k-1}, x_k, x_{k+1})$, and use them to write formulas for $\tilde{c}$,$\tilde{\alpha}$.

(b) Start with $x_0 = 0$ and carry out 15 iterations of $x_{k+1} = g(x_k)$ for each case. For each case, write an estimate of c, α based on part (a). In each case, use stability analysis for predict (or confirm) the values of c, α. The cases are:

i. $g(x) = (1 + 2x + 3x^3)/10$.

ii. $g(x) = (1 + 2x + 3x^3)/2$. Does this sequence converge?

iii. $g(x) = x - (2x + 3x^2 - 4)/(2 + 6x)$.

3. Analysis of a multistep method. Consider using the iterative method $x_{k+1} = g(x_k, x_{k-1})$ for finding a solution of $f(x) = 0$ with

$$g(x, y) = 2x - y - \frac{f(x)(x - y)}{f(x) - f(y)}$$

This is a modified version of the secand method. Is it a good method?

(a) Let $f(x) = x^2 - 3x + 2$, carry out the linear stability analysis for $x_k \to x^* = 2$. That is, let $x_k = 2 + e_k$ and find the eigenvalues and the general expression for the linearized error e_k.

(b) What can you conclude about the convergence of $\{x_k\}$ from the results of (a)?

(c) To test that, run the iterative scheme starting from $x_0 = 2.1$, $x_1 = 2.2$, and plot x_k versus k for $k = 1, 2, 3, \ldots, 100$. Does this agree with the predicted form for e_k for some range of k-values? What happens for large k's?

4. Newton's method. Write four routines to carry out versions of Newton's method for solving

$$f(x) = 2x + 3x^2 - 4 = 0$$

(a) Newton's method with the exact deriative function; call this `newton1`.

(b) Newton's method with a finite-difference derivative with dx (an input parameter); call this `newton2`.

(c) Newton-secant method with starting guess x_0 (an input parameter); call this `newton3`. You need to make some choice on what to do for calculating x_1 (not an input parameter) before you can start the "propoer secant method," describe what you use.

(d) The Regula Falsi method with starting guesses x_0, x_1 (both input parameters); call this `newton4`. Note: you do not need to keep the whole sequence $\{x_k\}$ for this method, at most you need to keep two previous values of x_k and their $f(x_k)$ values.

In all cases, minimize the number of function calls you use to make your routine as efficient as possible. In all four cases, you need to define some *stopping criterion*[1] for when you finite your iteration (either successfully [convergence] or unsuccessfully [divergence]). Describe what you use for this.

Then add a few lines of code to each of the routines to print out estimates of $\tilde{c}, \tilde{\alpha}$ before they stop. For `newton2()`, make a log-linear plot of dx versus α for the range $10^{-15} \leq dx \leq 0.1$. Explain the form of this graph, and suggest and optimal value for dx.

[1] Like Homework 9, problem 2.