# The Binomial Method for Option Pricing

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## 1 Introduction

An option is a contract between financial traders that gives the buyer the right, but not the obligation, to buy or sell a particular asset at a later time at a pre-determined price. In return for granting the option, the seller collects a payment from the buyer. They can be traded on a wide range of commodities and financial assets, such as stocks, currencies, bonds, corn, wheat, sugar, and petroleum. Options have been traded for centuries, but their popularity increased significantly with the introduction of a listed options exchange in 1973. Since then, options have become one of the most traded financial instruments in the market. As a result, the need for proper pricing of these instruments increased as well. Ironically, the main advance in this area was aslo achieved in 1973 when Myron Scholes and Fisher Black presented the fisrt completely satisfactory equilibrium option pricing model. Despite its many unrealistic assumptions, their model is still widely used in the financial industry. Moreover, in 1997 Scholes recieved the Noble Prize in Economics for his work on option pricing. Though ineligible for the prize because of his death in 1995, Black was also honored as a contributor during the ceremony.

Unfortunately, the Black-Scholes option pricing model employs advanced mathematics, which does not allow practioneers to get a complete grasp of the model. A more intuitive approach for option pricing was proposed by Cox, Rubenstein, and Ross in 1979. Their binomial method creates a binomial lattice of all possible price paths of the underlying asset, and then calculates the expected value that is shown to be the fair price of the option. It is not only easier to understand than the Black-Scholes model, but is also very accurate. In fact, Cox, Rubenstein, and Ross showed that if the number of time periods taken in the binomial method goes to infinity and the length of each time period is infinitesimally short, it will converge to the Black-Scholes model. Because of its simplicity and convergence, the binomial method has gained a lot of popularity in the financial industry.

## 2 The Payoffs from Using Options

The primary objective of all investors is to maximise returns and minimise risks. In other words, investors are willing to take on higher risks only if they are compensated with a higher return. Before we go into the theory of option pricing, it is essential to understand the risk and rewards that options provide. ptions can be grouped into two main categories depending on whether the their owner has the right to buy or to sell. A call option gives its holder the right to buy the underlying asset at a certain price (the strike price) on a certain date (the expiration date), whereas a put option gives the right to sell the underlying security. In order to present the payoff functions of calls and puts, we introduce the following notation:

K - the strike price of the option

T - the time of expiration

S - the value of the underlying asset

C-the value of a call option

P- the value of a put option

If the value of the asset is less than the strike price at the expiration date, the call option is not exercised and expires worthless. The investor loses the amount he paid to own the option (the premium). On the other hand, if the value of the asset is greater than the strike price the owner will buy the asset at the strike price. Then, he can sell the asset at the market price and realize a profit equal to the difference of the asset value and the strike price.

Therefore, the call option payoff can be expressed as:

C(S(T),T)= max( S(T)-K, 0)

If the value of the asset is less than the stike price at the expiration date, the holder of the put option will exersise it and make a profit equal to the difference between the strike price and the value of the asset. On the other hand, if the value of the asset is greater than the the strike price, the option will expire worthless and the investor will lose the the money he paid to own the option. therefore, the put option payoff is:

P(S(T),T)= min(K - S(T), 0)

The value of put and calls is related by what is called the put-call parity. Using the no arbitrage assumption, the put-call parirty states that at time t:

$C(t) + Ke^{-rt} = P(t) + S(t)$

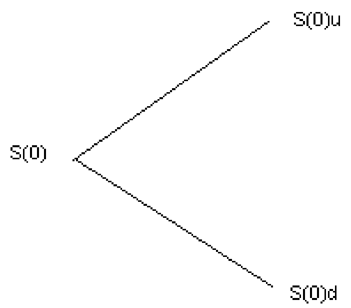
Thus, once we have found the value of the call, we can use it to calculate the price of the put.

# 3 The Binomial Method for Option Pricing

The binomial model is a simplified model of price movement on the underlying asset. It assumes that at each time step $t_i$, the price of the stock either goes up by a factor u (u $> 1$), or goes down by d ($0 \leq d \leq 1$). Moreover, the price of the stock during different increments $\Delta$ is independent. It also assumes that there are no arbitrage opportunities and that there is a riskless interest rate r which is available to everyone. By risk neutrality, we know that the expected return of the asset is equal to the value of the initial price of the asset invested in compounding interest. That is:

$E(S(t)) = S(0)e^{rt}$

The simplest model, involving only one time period, is shown in the next figure:



At time $t = 0$ the price of the asset is S(0), and at the next time step the there are two possibilities for the value of the asset: either goes up to S(0)u or goes down to S(0)d. For the model with n-steps:

$S(t_n) = S(n\Delta) = S(0)d^{n-k}u^k = S(0)d^n(\frac{u}{d})^k$

$\frac{S(i\Delta)}{S((i-1)\Delta)} = \begin{cases} uS((i-1)\Delta) & \text{with probability p} \\ dS((i-1)\Delta) & \text{with probability } 1-p \end{cases}$

We want to find u, d, and p in terms of the known quantities: the riskless interest rate r, the volatility of the underlying asset $\sigma^2$, and $\Delta$.

Using risk neutrality we have:

$E(\frac{S(\Delta)}{S(0)}) = up + d(1-p) = \frac{S(0)e^{r\Delta}}{S(0)} = e^{r\Delta}$

$up - pd = e^{r\Delta} - d$

$p = \frac{e^{r\Delta}-d}{u-d}$

This is our fisrt equation relating p, u, and d. We can also calculate the expected value of the squared return:

$E((\frac{S(\Delta)}{S(0)})^2) = u^2p + (1-p)d^2$

$E((\frac{S(\Delta)}{S(0)})^2) = u^2(\frac{e^{r\Delta}-d}{u-d}) + (\frac{u-d-e^{r\Delta}+d}{u-d})d^2$

$E((\frac{S(\Delta)}{S(0)})^2) = \frac{u^2e^{r\Delta}-u^2d+ud^2-e^{r\Delta}d^2}{u-d}$

$E((\frac{S(\Delta)}{S(0)})^2) = e^{r\Delta}(u+d) + ud$

The square of the expected return is:

$E(\frac{S(\Delta)}{S(0)})^2 = (e^{r\Delta})^2 = e^{2r\Delta}$

Since $\sigma^2\Delta = Var(\frac{S(\Delta)}{S(0)}) = E((\frac{S(\Delta)}{S(0)})^2) - E(\frac{S(\Delta)}{S(0)})^2$ we have that:

$\sigma^2\Delta = e^{r\Delta}(u+d) - ud - e^{2r\Delta}$

Thus, we have obtained two equations but we have three unknowns so we choose $u = \frac{1}{d}$ to be the third equation. Therefore:

$\sigma^2\Delta = e^{r\Delta}(u + \frac{1}{u}) - 1 - e^{2r\Delta}$

Let x = u or x = d. Thus,

$\sigma^2\Delta = e^{r\Delta}(x + \frac{1}{x}) - 1 - e^{2r\Delta}$

$1 + \sigma^2\Delta = e^{r\Delta}(x + \frac{1}{x}) - e^{2r\Delta}$

$e^{\sigma^2\Delta} = e^{r\Delta}(x + \frac{1}{x}) - e^{2r\Delta}$

Multiplying both sides of the equation by x we get:

$xe^{\sigma^2\Delta} = e^{r\Delta}(x^2 + 1) - xe^{2r\Delta}$

Multiplying both sides of the equation by $e^{-r\Delta}$ we get:

$x(e^{\sigma^2\Delta - r\Delta}) = x^2 + 1 - x(e^{2r\Delta - r\Delta})$

$x^2 + 1 - x(e^{r\Delta} + e^{\sigma^2\Delta - r\Delta}) = 0$

$x^2 + 1 - x(1 + r\Delta + 1 + \sigma^2\Delta - r\Delta) = 0$

$x^2 + 1 - x(e^{\sigma^2\Delta} + 1) = 0$

Using the quadratic formula we get:

$x = \frac{e^{\sigma^2\Delta}+1\pm\sqrt{(e^{\sigma^2\Delta}+1)^2-4}}{2}$

$x = \frac{e^{\sigma^2\Delta}+1\pm\sqrt{(1+2e^{\sigma^2\Delta}+e^{2\sigma^2\Delta}-4}}{2}$

$x = \frac{1+\sigma^2\Delta+\pm(1+2+2\sigma^2\Delta+1+2\sigma^2\Delta-4)}{2}$

$x = \frac{2+\sigma^2\Delta\pm\sqrt{4\sigma^2\Delta}}{2}$

$x = 1 + \frac{\sigma^2\Delta}{2} \pm \sigma\sqrt{\Delta}$

$x = 1 \pm \frac{(\sigma^2\Delta)^2}{2} \pm \sigma^2\Delta$

$x = e^{\pm\sigma\sqrt{\Delta}}$

Since u > 1 , we know that

$u = e^{\sigma\sqrt{\Delta}}$

Similarly, since d $(0 \leq d \leq 1)$ we have:

$d = e^{-\sqrt{\Delta}}$

Going back to the equation of p we have:

$p = \frac{e^{r\Delta}-d}{u-d} = \frac{e^{r\Delta}-e^{-\sigma\sqrt{\Delta}}}{e^{\sigma\sqrt{\Delta}}-e^{-\sigma\sqrt{\Delta}}}$

$p = \frac{1+r\Delta-1+\sigma\sqrt{\Delta}-\frac{\sigma^2\Delta}{2}}{1+\sigma\sqrt{\Delta}-1+\sigma\sqrt{\Delta}}$

$p = \frac{r\Delta+\sigma\sqrt{\Delta}-\frac{\sigma^2\Delta}{2}}{2\sigma\sqrt{\Delta}}$

$p = \frac{1}{2} + \frac{r\Delta-\frac{\sigma^2\Delta}{2}}{2\sigma\sqrt{\Delta}}$

$p = \frac{1}{2} + \frac{\Delta(r-\frac{\sigma^2}{2})}{2\sigma\sqrt{\Delta}}$

$p = \frac{1}{2} + \frac{\sqrt{\Delta}(r - \frac{\sigma^2}{2})}{2\sigma}$

$p = \frac{1}{2}(1 + \frac{\sqrt{\Delta}(r - \frac{\sigma^2}{2})}{\sigma})$

Now, let f denote the price of the option at the first time step. Then $f_u$ will be the value of the option if the price of the underlying asset went up and $f_d$ will be the value of the option if the price of the underlying asset went down. Consider a portfolio consisting of $\Delta$ units of the asset and $\psi$ units of the riskless asset so that it replicates the payoff of the option:

$\delta S(0)u + \psi e^{rt} = f_u$

$\delta S(0)u + \psi e^{rt} = f_d$

This is a system of two equation with two unknowns so we can find the solution:

$\Delta = \frac{f_u - f_d}{S(0)u - S(0)d}$

$\psi = e^{-rt} \frac{uf_d - uf_u}{u - d}$

Since the option payoff at time t is equal to that of the portfolio, the value of the portfolio must be equal to that of option. Thus at t=0 we must have:

$f = \Delta S(0) + \psi$

With the optimal values of $\Delta$and $\psi$ we get:

$f = S(0)\frac{f_u - f_d}{S(0)u - S(0)d} + e^{-rt}\frac{uf_d - uf_u}{u - d} = \frac{f_u - f_d + e^{-rt}(uf_d - uf_u)}{u - d}$

Note that $p = \frac{e^{r\Delta} - d}{u - d}$. Therefore, the value of the option at t=0 can be expressed as:

$f = e^{-rt}(pf_u + (1 - p)f_d)$

In a similar way, the binomial model can be applied to the same total period of time $t = n\Delta$. In that case:

$f = e^{-rt} \sum_{i=1}^{n} \binom{n}{i} p^i (1 - p)^{n-i} f(S(0)u^i d^{n-i})$

Moreover, Cox, Rubenstein, and Ross showed that the distribution of $\log \frac{S(n\Delta)}{S(0)}$ will approach the normal distribution as $\Delta \to 0$. To see that define:

$$Y(i) = \begin{cases} 1 & \text{if } S(i\Delta) = S((i-1)\Delta)u \\ \\ 0 & \text{if } S(i\Delta) = S((i-1)\Delta)d \end{cases}$$

Then,

$\sum_{i=1}^{n} Y(i) =$ number of times the price goes up

This is a sum of independent identically distributed random variables.

$n - \sum_{i=1}^{n} Y(i) =$ number of times the price goes down

Therefore, after n increments $\Delta$ the price of the security will be:

$$S(n\Delta) = S(0)(u)^{\sum_{i=1}^{n} Y(i)} (d)^{n - \sum_{i=1}^{n} Y(i)}$$

Taking the logarithm of both sides we get:

$$log(\frac{S(n\Delta)}{S(0)}) = log((u)^{\sum_{i=1}^{n} Y(i)} (d)^{n - \sum_{i=1}^{n} Y(i)})$$

$$log(\frac{S(n\Delta)}{S(0)}) = log((\frac{u}{d})^{\sum_{i=1}^{n} Y(i)} (d)^{n})$$

$$log(\frac{S(n\Delta)}{S(0)}) = nlod(d) + \sum_{i=1}^{n} Y(i) log(\frac{u}{d})$$

Since $n\Delta = t$ we know that $n = \frac{t}{\Delta}$. Thus,

$$log(\frac{S(n\Delta)}{S(0)}) = \frac{t}{\Delta} lod(d) + \sum_{i=1}^{\frac{t}{\Delta}} Y(i) log(\frac{u}{d})$$

But $u = e^{\sigma\sqrt{\Delta}}$ and $d = e^{-\sqrt{\Delta}}$ so:

$$log(\frac{S(n\Delta)}{S(0)}) = \frac{t}{\Delta}(-\sigma\sqrt{\Delta}) + \sum_{i=1}^{\frac{t}{\Delta}} Y(i)(\sigma\sqrt{\Delta} - (-(\sigma\sqrt{\Delta})))$$

As $\Delta \to 0$, $\frac{t}{\Delta} \to \infty$ and $\sum_{i=1}^{\frac{t}{\Delta}} Y(i)$ will be the infinite sum of independent identically distributed random variables. By the central limit theorem, it follows that $\sum_{i=1}^{\frac{t}{\Delta}} Y(i)$ is normally distributed. As a result, $log(\frac{S(n\Delta)}{S(0)})$ is normally distributed as $\Delta \to 0$. Moreover, we can calculate the parameters of the distribution:

$$E(log(\frac{S(n\Delta)}{S(0)})) = log(\frac{u}{d})np + nlog(d)$$

$E(log(\frac{S(n\Delta)}{S(0)})) = \frac{t}{\Delta}(-\sigma\sqrt{\Delta}) + \frac{t}{\Delta}p(2\sigma\sqrt{\Delta})$

$E(log(\frac{S(n\Delta)}{S(0)})) = -\frac{t}{\sqrt{\Delta}}(\sigma) + \frac{t}{\sqrt{\Delta}}p(2\sigma)$

$E(log(\frac{S(n\Delta)}{S(0)})) = -\frac{t\sigma}{\sqrt{\Delta}} + \frac{t\sigma}{\sqrt{\Delta}}(1 + \frac{u\sqrt{\Delta}}{\sigma})$

$E(log(\frac{S(n\Delta)}{S(0)})) = ut$

By independece we know that the variance will be:

$Var(log(\frac{S(n\Delta)}{S(0)})) = 4\sigma^2\Delta\sum_{i=1}^{\frac{t}{\Delta}} Y(i)$

$Var(log(\frac{S(n\Delta)}{S(0)})) = 4\sigma^2\Delta tp(1-p)$

$Var(log(\frac{S(n\Delta)}{S(0)})) = 4\sigma^2\Delta t(\frac{1}{2} + \frac{u\sqrt{\Delta}}{2\sigma})(\frac{1}{2} - \frac{u\sqrt{\Delta}}{2\sigma})$

$Var(log(\frac{S(n\Delta)}{S(0)})) = 4\sigma^2\Delta t(\frac{1}{4} - \frac{u^2\Delta}{4\sigma^2})$

$Var(log(\frac{S(n\Delta)}{S(0)})) = \sigma^2 t - u^2 t\Delta$

As $\Delta \to 0$,

$Var(log(\frac{S(n\Delta)}{S(0)})) = \sigma^2 t$

## 4 The Black -Scholes Option pricing Model

Myron Scholes and Fisher Black were able to show that under certain assumptions, the price of the option will satisfy the following partial differential equation:

$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial t^2} - rf + \frac{\partial f}{\partial t} = 0$

The assumptions of their model are the following:

(1) The price of the underlying security follows a geometric brownian motion.

(2) There are no arbitrage opportunities in the market.

(3) There are no transaction costs on trading.

(4) The underlying asset does not pay any dividends.

(5) Trading is continious.

Black and Scholes were able to solve the partial differential equation and arrived at the following result:

$C = SN(d_1) - Ke^{-rt}N(d_2)$ where

$d_1 = \frac{\ln(\frac{S}{K})+(r+\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$

$d_2 = d_1 - \sigma\sqrt{t}$ and

$N(d_1)$ is the cumulative standard normal probability that the value is less than $d_1$.

Using the put-call parity, the value of the put is found to be:

$P = Ke^{-rt}N(-d_2) - SN(-d_1)$

# 5 Convergence of the Binomial method

Cox, Ross and Rubinstein showed that if the number of time periods taken in the binomial method goes to infinity and the length of each time period is infinitesimally short, it will converge to the Black-Scholes model. Their proof, however, is unnecessarily long and relies on a specific case of the Central Limit Theorem. A more general proof of the convergence of the binomial to the Black-Scholes model is provided by Hsia. Here is an outline of his proof. The price of a call under the binomial method is:

$$C = e^{-rt}\sum_{i=1}^{n}\binom{n}{i}p^i(1-p)^{n-i}max[0, u^i d^{n-i}S - K]$$

Note that for some outcomes $max[0, u^i d^{n-i}S - K]$ is zero. Let $\alpha$ represent the minimum number of upward moves for the call so that the call is excercisable at the end. In other words, $\alpha$ is the smallest integer such that $u^\alpha d^{n-\alpha}S > K$. Then for all $i < \alpha, max[0, u^i d^{n-i}S - K] = 0$ and for $i \geq \alpha, max[0, u^i d^{n-i}S - K] = u^i d^{n-i}S - K$. Thus, the value of the call can be expressed as:

$$C = e^{-rt}\sum_{i=\alpha}^{n}\binom{n}{i}p^i(1-p)^{n-i}u^i d^{n-i}S - K$$

This can be rewritten as:

$$C = Se^{-rt}\sum_{i=\alpha}^{n}\binom{n}{i}p^i(1-p)^{n-i}u^i d^{n-i} - Ke^{-rt}\sum_{i=\alpha}^{n}\binom{n}{i}p^i(1-p)^{n-i}u^i$$

Let $B_1 = e^{-rt}\sum_{i=\alpha}^{n}\binom{n}{i}p^i(1-p)^{n-i}u^i d^{n-i}$ and

$B_2 = \sum_{i=\alpha}^{n}\binom{n}{i}p^i(1-p)^{n-i}u^i$ Then,

$C = SB_1 - Ke^{-rt}B_2$

In order to show convergence to the Black-Scholes formula we must show that $B_1$ and $B_2$ converge to $N(d_1)$ and $N(d_2)$ repectively. Since we require that $u^\alpha d^{n-\alpha}S > K$ we have that:

$\alpha\log(u) + (n-\alpha)\log(d) + \log(S) > \log(K)$

$\alpha\log(u) + n\log(d) - \alpha\log(d) + \log(S) > \log(K)$

$\alpha(\log(u) - \log(d)) > \log(K) - \log(S) - n\log(d)$

$\alpha > \frac{\log(\frac{K}{S}) - n\log(d)}{log(\frac{u}{d})}$

Since $\alpha$ needs to be an integer we write:

$\alpha = \frac{\log(\frac{K}{S}) - n\log(d)}{log(\frac{u}{d})} + \zeta$

where $\zeta$ is a number added to our computed number to make $\alpha$ an integer. In the limit, $\zeta$ will converge to zero as there will be an infinite number of integer steps.

By the DeMoivre-LaPlace limit theorem we know that a binomial distribution will converge to the normal if np $\to\infty$ as n $\to 0$. Thus, for $B_1$ we need:

$B_1 \to \int_\alpha^b f(i)\,di.$

where f(i) is the density for a normal distribution. However, f(i) is not standard normal so we must convert it by defining $z = \frac{i-E[i]}{\sigma_i}$. Hsia, however, defines $d = \frac{-(i-E[i])}{\sigma_i}$ so that:

$B_1 \to \int_\alpha^\infty f(i)\,di. = \int_-\infty^d f(z)\,dz. = N(d)$

Since $E(log(\frac{S(n\Delta)}{S(0)})) = E[i]\log(\frac{u}{d}) + n\log(d)$ we have that:

$E[i] = \frac{E(log(\frac{S(n\Delta)}{S(0)})) - n\log(d)}{\log(\frac{u}{d})}$

Similarly, since $Var(log(\frac{S(n\Delta)}{S(0)})) = Var[i](\log(\frac{u}{d}))^2$ we have that:

$Var[i] = \frac{Var(log(\frac{S(n\Delta)}{S(0)}))}{(\log(\frac{u}{d}))^2}$

Therefore,

$$d = \frac{\frac{\log(\frac{S}{K})+E(log(\frac{S(n\Delta)}{S(0)}))}{log(\frac{u}{d})}-\zeta}{\frac{\sqrt{Var(log(\frac{S(n\Delta)}{S(0)}))}}{\log(\frac{u}{d})}}$$

From the properties of the binomial distribution we know that $Var[i] = np(1-p)$ Thus,

$$d = \frac{\log(\frac{S}{K})+E(log(\frac{S(n\Delta)}{S(0)}))}{\sqrt{Var(log(\frac{S(n\Delta)}{S(0)}))}} - \frac{\log(\frac{u}{d})\zeta}{\sqrt{np(1-p)}}$$

As $n \to \infty$, the second term will go to zero. Also, we have already showed that $Var(log(\frac{S(n\Delta)}{S(0)})) = \sigma^2 t$. Thus, we have:

$$d = \frac{\log(\frac{S}{K})+E(log(\frac{S(n\Delta)}{S(0)}))}{\sigma\sqrt{t}}$$

We need this to equal $d_1$ and $d_2$ as defined by the Black-Scholes formula. This means that we need:

$$E(log(\frac{S(n\Delta)}{S(0)})) = \begin{cases} (r+\frac{\sigma^2}{2})t & \text{if the probability is } p^* \\ (r-\frac{\sigma^2}{2})t & \text{if the probability is p} \end{cases}$$

Note that $E(log(\frac{S(0)}{S(n\Delta)})) = \prod_{i=1}^{n} E[\frac{(S((i-1)\Delta)}{S(i\Delta)}]$, since the variables are independent. Also,

$$E(log(\frac{S(0)}{S(n\Delta)})) = p^*(\frac{1}{u}) + (1-p^*)(\frac{1}{d})$$

Therefore,

$$E(log(\frac{S(0)}{S(n\Delta)})) = \prod_{i=1}^{n} p^*(\frac{1}{u}) + (1-p^*)(\frac{1}{d})$$

$$E(log(\frac{S(0)}{S(n\Delta)})) = [p^*(\frac{1}{u}) + (1-p^*)(\frac{1}{d})]^n$$

Inverting this we get:

$$[E(log(\frac{S(0)}{S(n\Delta)}))]^{-1} = [p^*(\frac{1}{u}) + (1-p^*)(\frac{1}{d})]^{-n}$$

Since $e^{rt} = [p^*(\frac{1}{u}) + (1-p^*)(\frac{1}{d})]^{-n}$ we have that:

$$-tr = \log(E(log(\frac{S(0)}{S(n\Delta)}))$$

Since $\frac{S(0)}{S(n\Delta)}$ is lognormally distributed we have:

$$-tr = E(log(\frac{S(0)}{S(n\Delta)})) + \frac{Var[(\frac{S(0)}{S(n\Delta)})]}{2}$$

$$-tr = -E(log(\frac{S(n\Delta)}{S(0)})) + \frac{Var[(\frac{S(n\Delta)}{S(0)})]}{2}$$

Now, since $Var(log(\frac{S(n\Delta)}{S(0)})) = \sigma^2 t$ we have:

$E(log(\frac{S(n\Delta)}{S(0)})) = (r + \frac{\sigma^2}{2})t$

Therefore, $B_1$ will converge to $N(d_1)$

To show that $B_2$ will converge to $N(d_2)$, note that:

$E(log(\frac{S(n\Delta)}{S(0)})) = \prod_{i=1}^{n} E[\frac{S(i\Delta)}{S((i-1)\Delta)}] = \prod_{i=1}^{n} pu + (1 - pd)$

$E(log(\frac{S(n\Delta)}{S(0)})) = (pu + (1 - pd))^n = e^{rt}$

Therefore,

$[\frac{S(n\Delta)}{S(0)}] = tr$

Once again, using the fact that $\frac{S(n\Delta)}{S(0)}$ is lognornally distributed we have:

$tr = E[log(\frac{S(n\Delta)}{S(0)})] + \frac{Var[(\frac{S(n\Delta)}{S(0)})]}{2}$

Thus,

$E[log(\frac{S(n\Delta)}{S(0)})] = (r + \frac{\sigma^2}{2})t$

This shows that $B_2$ will converge to $N(d_2)$ and so the binomial method will converge to the Black-Scholes model if the time periods taken goes to infinity and the length of each time period goes to zero.

# 6 Application of The Binomial Method to a Real World Problem

Rigby oil [1] owns the drilling rights for a small oil field in the North Sea for the next five years. However, it still hasn't started extracting oil and has just received a offer of \$20 million from its competitor McKensey Oil for the drilling rights in their entirety.The estimated oil reserve in the oil field 1 milliion barrels. The current price of oil is \$56 per barrel and extraction costs are \$50 per barrel. Moreover, the 4-year riskless rate is 6.25 % and the volatility of the oil price in the market is 40 %. In order, to make a decision Rigby Oil may choose use the binomial method.

[1]Based on Emma Rasiel's notes for the Asset Pricing and Risk Management class at Duke University

In essence, Rigby oil currently owns a call option and has to figure out whether its value is higher than the $20 million offered by its competitor. The inputs of the binomial method are:

$S(0) = 56$

$K = 50$

$T = 4$

$r = 0.0625$

$\sigma = 0.4$

Using four steps for the calculation, we find that the value of the call option is $ 24.49.[2]Since Rigby Oil can extract 1 million barrels, the value of its options is $ 24.49 millions. This is higher than the offer from McKensey Oil, so the company should decline the offer.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[2]A matlab code for the calculation of the call price, based on Kevin Cheng's code from http://www.global-derivatives.com, is provided on the last page

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