Math 107: Linear Algebra and Differential Equations

Test #2

Name: Answer Key *Thursday, November 19, 2009*

Lecture section: 107.0__________ Recitation section: 107R.0__________

All answers must be justified. No calculator is allowed.

Question 1. [5 points]

Let

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

From $A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \vec{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Question 2. [15 points]
The night before Thanksgiving, I realized to my horror that I had forgotten to defrost my turkey. So I hurried up and took the frozen turkey from the freezer at $-5°C$ into a room with temperature $20°C$. After 30 minutes, the temperature of the turkey is $-2°C$.

(a) Describe a model for this problem, based on Newton's law of cooling, which states that the rate of change of temperature of an object is proportional to the difference between its own temperature and the ambient temperature (i.e., the temperature of its surroundings). Include a differential equation, and one or more conditions on the solution. State clearly the unit of each variable.

$$\frac{dT(t)}{dt} = k[T(t) - 20]$$

T in $°C$
k in 1/minute

initial condition: $T(0) = -5$

other condition $T(30) = -2$

(b) How long will it take the turkey to reach $0°C$?

Homo. soln, $T_H(t) = C_1 e^{kt}$
particular soln $T_p(t) = C_2 = 20$

IC: $-5 = T(0) = C_1 + 20 \Rightarrow C_1 = -25$

Determine k:

$$-2 = T(30) = -25e^{30k} + 20 \Rightarrow e^{30k} = \frac{22}{25}$$

$$\Rightarrow k = \frac{1}{30} \ln\left(\frac{22}{25}\right)$$

Solve for time $T(t) = 0$

$$0 = T(t) = -25e^{kt} + 20$$

$$\Rightarrow t = \frac{1}{k} \ln\left(\frac{20}{25}\right) = 30 \frac{\ln(20/25)}{\ln(22/25)}$$

Question 3. [20 points]
Find an orthonormal basis for the plane U with equation $x + y + z = 0$.
Hint: You might want to first come up with *a* basis for U, orthonormal or not. And then, starting with that basis, use the technique described in class to construct an orthonormal basis.

A basis for U is for example

$$\left\{ u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Apply Gram-Schmidt to $\{u_1, u_2\}$:

$$v_1 = \frac{u_1}{\|u_1\|} = \boxed{\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}}$$

Let $w_2 = u_2 - \langle u_2, v_1 \rangle v_1$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$v_2 = \frac{w_2}{\|w_2\|} = \boxed{\sqrt{\frac{2}{3}} \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}}$$

$\{v_1, v_2\}$ is an orthonormal basis for U.

Question 4. [20 points] Consider

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$$

(a) Find all the eigenvalues and the corresponding eigenvectors of A.

$$\det(A - \lambda I) = \det\begin{bmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{bmatrix} = -(4-\lambda)(\lambda+1) - 6 = 0$$

$$\Rightarrow \lambda = 1, 2$$

$$\lambda = 1 \Rightarrow v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 2 \Rightarrow v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

(b) A is similar to a diagonal matrix D, i.e.,

$$A = PDP^{-1}$$

Find P and D.

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

Question 4. (Continued)

(c) Compute A^6 without actually multiplying A with itself.

$$A^6 = (PDP^{-1})^6 = PD\cancel{P^{-1}}\cancel{P}D\cancel{P^{-1}}\ldots DP^{-1}$$

$$= PD^6P^{-1}$$

$$D^6 = \begin{bmatrix} 1 & 0 \\ 0 & 64 \end{bmatrix}$$

$$A^6 = \begin{bmatrix} 190 & -189 \\ 126 & -125 \end{bmatrix}$$

Question 5. [20 points]

(a) Find the homogeneous solution to

$$3\frac{d^2y}{dx^2}(x) + 6\frac{dy}{dx}(x) - 24y(x) = 0$$

char eq: $3\lambda^2 + 6\lambda - 24 = 0$

$\Rightarrow \lambda = 2, -4$

$y_H(x) = c_1 e^{2x} + c_2 e^{-4x}$

(b) Find the general solution to

$$3\frac{d^2y}{dx^2}(x) + 6\frac{dy}{dx}(x) - 24y(x) = -5e^{2x} - 6\sin(4x)$$

$y_{P1} = Axe^{2x}$

plug in: $-5e^{2x} = 3(4Ae^{2x} + 4Axe^{2x}) + 6(Ae^{2x} + 2Axe^{2x}) - 24Axe^{2x} = 18Ae^{2x}$

$\Rightarrow A = -\frac{5}{18}, \quad y_{P1} = -\frac{5}{18}xe^{2x}$

$y_{P2} = c_1 \cancel{4} \sin(4x) + c_2\cos(4x)$

plug in: $-6\sin(4x) = 3(-16c_1\sin 4x - 16c_2\cos 4x)$
$+ 6(4c_1\cos 4x - 4c_2\sin 4x)$
$- 24(c_1\sin 4x + c_2\cos 4x)$
$= (-72c_1 - 24c_2)\sin 4x + (24c_1 - 72c_2)\cos 4x$

so $c_1 = \frac{3}{40}, \quad c_2 = \frac{1}{40}, \quad y_{P2} = \frac{3}{40}\sin 4x + \frac{1}{40}\cos 4x$

$y = y_H + y_{P1} + y_{P2}$

Question 6. [15 points] Consider the mass-spring system

$$m\frac{dy^2}{dt^2} + \omega\frac{dy}{dt} + ky = f$$

(a) Suppose $m = 2$ kg, $\omega = 4$ kg/s and $k = 1.5$ kg/s^2. Is this system overdamped, underdamped, or critically damped?

$$\omega^2 - 4mk = 16 - 12 = 4 > 0$$

overdamped

(b) If $m = 2$ kg and $f = 20$ m/s^2, find the general solution of the mass-spring system.

$$2y'' + 4y' + \frac{3}{2}y = 20$$

Char eq: $2\lambda^2 + 4\lambda + \frac{3}{2} = 0 \Rightarrow r = -\frac{1}{2}, -\frac{3}{2}$

$$y_H = c_1 e^{-t/2} + c_2 e^{-3t/2}$$

$y_p = c$ plug in $\Rightarrow$ $c = 40/3$

$$y(t) = c_1 e^{-\frac{t}{2}} + c_2 e^{-\frac{3t}{2}} + \frac{40}{3}$$

(c) If $m = 25/6$ kg, $\omega = 0$, and $f = \cos(\nu t)$, what value of ν will result in resonance?

$$\frac{25}{6} y'' + \frac{3}{2} y = \cos wt$$

Char eq: $\frac{25}{6}\lambda^2 + \frac{3}{2} = 0 \Rightarrow r = \pm i\frac{3}{5}$

Resonance $\nu = \frac{3}{5}$