Math 107: Linear Algebra and Differential Equations

Test #1

Name: ______________________________ *Thursday, October 1, 2009*

All answers must be justified.

Question 1. [10 points]

(a) Find the inverse of the following matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

(b) Use the inverse to compute the solution to the system of equations

$$\begin{aligned} x - 2y &= 5 \\ 2x - 3y &= 7 \end{aligned}$$

$x = -1$
$y = -3$

Question 2. [16 points] Suppose that A and B are two $n \times n$ matrices with $\det(A) = 3$ and $\det(B) = 7$. Compute the following:

(a) $\det(ABA) =$

$3 \times 7 \times 3 = 63$

(b) $\det(A^{-1}) =$

$1/3$

(c) $\det(BAB^{-1}) =$

$7 \times 3 \times 1/7 = 3$

(d) $\det(A^T B^{-1}) =$

$3/7$

(e) $\det(2A) =$

$2^n \cdot 3$

Question 3. [20 points] Consider the system of questions with unknowns x_1, x_2, x_3:

$$\begin{aligned} ax_1 + x_3 &= 1 \\ 2(b-a)x_2 + 2x_3 &= 4 \\ 2ax_1 + (b-a)x_2 + x_3 &= 2 \end{aligned}$$

(a) Determine the conditions on a and b so that the above system of equations admits a unique solution.

$$\left[\begin{array}{ccc|c} a & 0 & 1 & 1 \\ 0 & \cancel{2}(b-a) & \cancel{2} & \cancel{4} \\ 2a & b-a & 1 & 2 \end{array}\right] \xrightarrow{2} \left[\begin{array}{ccc|c} a & 0 & 1 & 1 \\ 0 & b-a & 1 & 2 \\ \cancel{2}a & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & b-a & 1 & 2 \\ a & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & b-a & 0 & 1 \\ a & 0 & 0 & 0 \end{array}\right]$$

$b \neq a$ and $a \neq 0$

(b) Determine the conditions on a and b so that the above system of equations has no solution.

$b = a$

Question 4. [10 points] Under what condition on the constants a and b is the vector

$$\begin{bmatrix} a \\ 1 \\ b \end{bmatrix}$$

in the linear span of the vectors:

$$\begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 9 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & a \\ 1 & 2 & 1 & 1 \\ 7 & 4 & 9 & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 3 & a \\ 7 & 4 & 9 & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & 1 & a-2 \\ 0 & -10 & 2 & b-7 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & 1 & a-2 \\ 0 & 0 & 0 & b-7-2(a-2) \end{array}\right]$$

$$b-7-2(a-2)=0$$

or $2a-b+3=0$

Question 5. [14 points] Given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find the rank and the dimension of the null space of A.

$$A \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

One free variable $\to$ $\dim(NS(A)) = 1$.

Two pivots $\to$ $\text{rank}(A) = 2$.

Question 6. [15 points]
(a) Show that the set

$$U = \left\{ \begin{bmatrix} x \\ x-y \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

satisfies the conditions for being a subspace of $\mathbb{R}^3$.

① It contains $\vec{0}$ ($x=0$, $y=0$)

② closed under addition

if $u = \begin{bmatrix} x_1 \\ x_1 - y_1 \\ y_1 \end{bmatrix}$ and $v = \begin{bmatrix} x_2 \\ x_2 - y_2 \\ y_2 \end{bmatrix}$ then $u+v = \begin{bmatrix} x_1 + x_2 \\ (x_1+x_2) - (y_1+y_2) \\ y_1 + y_2 \end{bmatrix}$

③ closed under scalar multiplication

$$\alpha u = \begin{bmatrix} \alpha x_1 \\ \alpha(x_1 - y_1) \\ \alpha y_1 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_1 - \alpha y_1 \\ \alpha y_1 \end{bmatrix}$$

(b) Find a basis for U. What is the dimension of U?

$$\begin{bmatrix} x \\ x-y \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{basis} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

dimension $= 2$

Question 7. [20 points] (a) Determine whether

$$V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ -1 \end{bmatrix} \right\}$$

form a spanning set for $\mathbb{R}^3$. Explain your answer.

Find c_1, c_2, c_3 so that $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ 10 \\ -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 7 & a \\ 2 & 3 & 10 & b \\ 3 & 5 & -1 & c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 7 & a \\ 0 & 5 & -4 & b-2a \\ 0 & 8 & -22 & c-3a \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 7 & a \\ 0 & 5 & -4 & b-2a \\ 0 & 0 & -15.6 & c-3a-(b-2a)\times\frac{5}{8} \end{array}\right]$$

system has a unique solution

V form a spanning set for $\mathbb{R}^3$.

(b) Does V above form a basis for $\mathbb{R}^3$? Why or why not.

Yes because there are 3 vectors in V.