Math 107: Linear Algebra and Differential Equations

Practice Test #2

Name: ____ *Thursday, November 19, 2009*

Lecture section: 107.0____________ Recitation section: 107R.0____________

All answers must be <u>justified</u>. No calculator is allowed.

Question 1. Find an orthonormal basis for the space spanned by the vectors

$$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

The first unit vector is $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \Big/ \left\| \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\| = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

A vector ~~that~~ in the span of the two given vectors and orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is

$$\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}\right)\right) = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} 2 = \begin{bmatrix} 0 \\ -3 \\ 4 \end{bmatrix}$$

A unit vector in that direction is

$$\begin{bmatrix} 0 \\ -3 \\ 4 \end{bmatrix} \Big/ \left\| \begin{bmatrix} 0 \\ -3 \\ 4 \end{bmatrix} \right\| = \begin{bmatrix} 0 \\ -3/5 \\ 4/5 \end{bmatrix}$$

Question 2. Find the general solution of

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 8xe^{2x}\cos x$$

Char eq for $y'' - 4y + 5y = 0$

is $\lambda^2 - 4\lambda + 5 = 0 \Rightarrow \lambda = 2 \pm i.$

Thus $y_H = e^{2x}(c_1 \cos x + c_2 \sin x)$

$$y_P = xe^{2x}(A_0 \cos x + B_0 \sin x) + x^2 e^{2x}(A_1 \cos x + B_1 \sin x)$$

Substituting into diff eq yields

$$(-2A_0 + 2B_1 - 4x A_1)\sin x + (2B_0 + 2A_1 + 4x B_1)\cancel{\sin}\cos x = 8x \cos x$$

So
$$\left.\begin{aligned} -2A_0 + 2B_1 &= 0 \\ 2A_1 + 2B_0 &= 0 \\ -4A_1 &= 0 \\ 4B_1 &= 8 \end{aligned}\right\} \quad \begin{aligned} A_1 = B_0 &= 0 \\ A_0 &= 2 \\ B_1 &= 0 \end{aligned}$$

So $y_P = 2xe^{2x}(x \sin x + \cos x)$

General soln

$$y = e^{2x}\big(c_1 \cos x + c_2 \sin x + 2x(x \sin x + \cos x)\big)$$

Question 3. A 2-kg mass stretches a spring 1 m. This mass is hung vertically on the spring and then a shock absorber is attached that exerts a resistance of 14 kg/s to the motion. The mass is pulled down 3 m and then released.

(a) Determine the motion of the mass.

$$mg = kL_0 \Rightarrow 20 = k \cdot 1 \Rightarrow k = 10$$

Eq. $2x'' + 14x' + 20x = 0$, $x(0) = 3$

Char eq: $2\lambda^2 + 14\lambda + 20 = 0 \Rightarrow \lambda = -2, -5$

$$x(t) = c_1 e^{-2t} + c_2 e^{-5t}$$

$$\left.\begin{aligned} x(0) &= c_1 + c_2 = 3 \\ x'(0) &= -2c_1 - 5c_2 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 5 \\ c_2 &= -2 \end{aligned}$$

so $x(t) = 5e^{-2t} - 2e^{-5t}$

(b) Determine the motion of the mass if an external force of $6e^{-2t}$ N is applied to the mass-spring system above.

$$20x'' + 14x' + 20 = 6e^{-2t}$$

let $x_p = Ate^{-2t} \Rightarrow x_p' = Ae^{-2t}(1-2t)$

$$x_p'' = Ae^{-2t}(-4+4t)$$

Subs into eq and cancel out e^{-2t}:

$$A\{2(-4+4t) + 14(1-2t) + 20t\} = 6$$

t terms all vanish, good.

$$(-8+14)A = 6 \Rightarrow A = 1.$$

$$x(t) = c_1 e^{-2t} + c_2 e^{-5t} + te^{-2t}$$

$$IC \Rightarrow c_1 = \frac{14}{3}, \quad c_2 = \frac{-5}{3}$$

Question 4. Find all the eigenvalues of

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

And for each eigenvalue, find a basis for the corresopnding eigenspace.

$$A - \lambda I = 0 \Rightarrow \begin{bmatrix} 2-\lambda & -1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0$$

$$\det(A - \lambda I) = 0 \Rightarrow (\lambda-3)(\lambda-1)^2 = 0$$

so $\lambda = 3, 1, 1.$

E-vec:

$\lambda = 3$: Subs into $(A - \lambda I)V = 0$ yields

$$-V_1 - V_2 = 0$$
$$V_1 - 3V_2 = 0$$
$$\Rightarrow \vec{V} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \quad \text{for } r \neq 0$$

$\lambda = 1$:

$$V_1 - V_2 = 0$$
$$2V_3 = 0$$
$$\Rightarrow \vec{V} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} s \quad s \neq 0$$

Question 5. Consider the linear transformation $T : P_2 \to P_2$ defined by

$$T(ax^2+bx+c) = (a+b+c)x^2 + (2a+3b+c)x + (3a+5b+c).$$

Find the kernel and range of T. Find their dimensions.

Kernel: Find null space of

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 1 \end{bmatrix} \overset{\text{reduce}}{\sim} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x = r\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad r = \text{free variable.}$$

$$\text{Ker}(T) = \left\{ \vec{x} = r\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, r \in \mathbb{R} \right\}$$

dim = 1

Range: Find $\vec{y}$ such that

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 1 \end{bmatrix} \vec{x} = \vec{y} \text{ is consistent}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 2 & 3 & 1 & y_2 \\ 3 & 5 & 1 & y_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 0 & 1 & -1 & y_2 - 2y_1 \\ 0 & 0 & 0 & y_1 - 2y_2 + y_3 \end{array}\right]$$

$$\Rightarrow y_1 - 2y_2 + y_3 = 0$$

$$\text{Range}(T) = \left\{ \vec{y} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, s, t \in \mathbb{R} \right\}.$$

dim = 2

Question 6. If T is defined on $\Re^2$ by

$$T(x_1, x_2) = (x_1 + 3x_2, 4x_1 + 2x_2)$$

find the matrix A of T relative to the basis

$$\beta = \left\{ \underset{v_1}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 3 \\ 5 \end{bmatrix}} \right\}$$

$$T(v_1) = \begin{bmatrix} 2+3\cdot 3 \\ 4.2+2\cdot 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 14 \end{bmatrix}, \quad T(v_2) = \begin{bmatrix} 18 \\ 22 \end{bmatrix}$$

Find $[T(v_1)]_\beta$ and $[T(v_2)]_\beta$ by solving

$$c_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 14 \end{bmatrix} \text{ and } k_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + k_2\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 18 \\ 22 \end{bmatrix}$$

Solutions are

$$[T(v_1)]_\beta = \begin{bmatrix} 13 \\ -5 \end{bmatrix} \text{ and } [T(v_2)]_\beta = \begin{bmatrix} 24 \\ -10 \end{bmatrix}$$

Thus $$A = \begin{bmatrix} 13 & 24 \\ -5 & -10 \end{bmatrix}$$