Math 107: Linear Algebra and Differential Equations

Mock Exam #1

Name: Smart Student — *Thursday, October 1, 2009*

All answers must be justified.

Question 1. Compute the determinant and the inverse matrix of

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\det(A) = -1 \cdot \det\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = -1 \cdot (2-1) = \boxed{-1}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & -2 & -1 & 1 & 0 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 2 & 0 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 2 & 0 \end{bmatrix}}$$

Question 2. The $n \times n$ matrix A can be turned into the identity matrix by swapping twos 3 and 4, multiplying row 2 by -3, subtracting row 5 from both rows 1 and 2, swapping rows 2 and 3, and the multiplying row 1 by 1/7 and row 3 by 2/3. What is the determinant of A?

$$\det(A) \cdot (-1) \cdot (-3) \cdot (-1) \cdot \left(\tfrac{1}{7}\right)\left(\tfrac{2}{3}\right) = \det(I)$$
$$\det(A) \cdot \left(-\tfrac{2}{7}\right) = 1$$
$$\det(A) = -\tfrac{7}{2}$$

Question 3. For what values of k and h is the following system consistent?

$$\begin{aligned} 2x_1 - x_2 &= h \\ -6x_1 + 3x_2 &= k \end{aligned}$$

Replace ~~row~~ Eq. 2 by ~~row~~ Eq. 2 + 3 × Eq. 1 :

$$\begin{aligned} 2x_1 - x_2 &= h \\ 0 &= k + 3h \end{aligned}$$

The system has no solution if $k+3h \neq 0$.
The system is consistent for any values of h and k that make $k+3h=0$.

Question 4. Determine by inspection if the given set if linearly dependent. Explain why.

(a). $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$

The set contains 4 vectors that are in $\mathbb{R}^3$.
Since $\dim(\mathbb{R}^3) = 3$, the set is linearly dependent.

(a). $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$

Because the zero vector is in the set, the set is linearly dependent.

(a). $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

Comparing the first 3 corresponding entries suggest that the second vector is $-3/2 \times$ the first vector. But this relation fails to hold for the last entry. So they are linearly independent.

Question 5. Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad w = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

(a) Is w in $\{v_1, v_2, v_3\}$? How many vectors are in $\{v_1, v_2, v_3\}$?

w is <u>not</u> in $\{v_1, v_2, v_3\}$.

There are only 3 vectors in $\{v_1, v_2, v_3\}$.

(b) How many vectors are in span$\{v_1, v_2, v_3\}$?

Infinitely many.

(c) Is w in the subspace spanned by $\{v_1, v_2, v_3\}$?

Find c_1, c_2, c_3 so that

$$w = c_1 v_1 + c_2 v_2 + c_3 v_3$$

or $$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = 1, \ c_2 = 1, \ c_3 = 0$$

or $w = v_1 + v_2$

so w is in the subspace spanned by $\{v_1, v_2, v_3\}$

Question 6. Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

(a) Determine if u is in the null space of A. Could u be in the column space of A?

$$Au = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow u$ is not a solution of $AX=0$, and u is not in $NS(A)$.

Also, with 4 entries, u cannot be in $CS(A)$ which is a subspace of $\mathbb{R}^3$.

(b) Determine if v is in the column space of A. Could v be in the null space of A?

Reduce $[A \vdots v]$ to an echelon form

$$[A \vdots v] = \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & 2 \\ 0 & 0 & 0 & 17 & 1 \end{array}\right]$$

So $AX=V$ is consistent so V is in $CS(A)$.

With only 3 entries V cannot be in $NS(A)$ which is a subspace of $\mathbb{R}^4$.

Question 7. Let H be the set of points inside and on the unit circle in the xy-plane. That is, let

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$$

Find a specific example—two vectors, or a vector and a scalar—to show that H is not a subspace of $\mathbb{R}^2$.

Take $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ which is in H.

Take $c = 2$, which is in $\mathbb{R}$.

$c \begin{bmatrix} x \\ y \end{bmatrix} = 2 \times \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. But $1^2 + 1^2 = 2 > 1$

So H is not closed under scalar multiplication and therefore not a subspace of $\mathbb{R}^2$.

Alternative, I can consider

$$\begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and show that H is not closed under addition.