Math 107: Linear Algebra and Differential Equations

Practice final exam

Name: ______________________________ *Friday, December 11, 2009*

Lecture section: 107.0__________ Recitation section: 107R.0__________

All answers must be justified. No calculator is allowed.

Question 1. Find conditions on a, b, c so that $v = (a, b, c)$ in $\Re^3$ belongs to $W = \text{span}(u_1, u_2, u_3)$, where

$$u_1 = (1, 2, 0), \quad u_2 = (-1, 1, 2), \quad u_3 = (3, 0, -4)$$

Set $V = xu_1 + yu_2 + zu_3$

$\Rightarrow (a, b, c) = x(1,2,0) + y(-1,1,2) + z(3,0,-4)$

$= (x - y + 3z,\ 2x + y,\ 2y - 4z)$

Equivalent system of linear equations:

$x - y + 3z = a$
$2x + y = b$
$2z - 4 = 0$

rewrite in matrix form, then reduce to echelon form:

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 0 & 3 & -6 & b-2a \\ 0 & 0 & 0 & 4a-2b+3c \end{array}\right]$$

Condition: $4a - 2b + 3c = 0$

Question 2. Find the general solution of the system

$$\begin{aligned} x_1' &= 2x_2 + 2x_3 \\ x_2' &= 2x_1 + 2x_3 \\ x_3' &= 2x_1 + 2x_2 \end{aligned}$$

char eq is

$$\det\begin{bmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{bmatrix} = 0 \quad \Rightarrow -(\lambda+2)^2(\lambda-4)=0$$

$$\lambda = -2, 4$$

For $\lambda = 4$, $\quad V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = -2$ $\quad V_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\quad V_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

$$x(t) = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{4t} + c_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{-2t} + c_3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{-2t}$$

Question 3. Find a basis and dimension of the subspace W of $\Re^3$ where

(a) $W = \{(a, b, c) : a + b + c = 0\}$

(Note $W \neq \mathbb{R}^3$. e.g. $(1,2,3) \notin W$, so $\dim W < 3$.)

Letting b and a be free var in $a+b+c=0$, we find

$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are two independent vectors in W.

Thus $\dim W = 2$, and u_1, u_2 form a basis of W.

(a) $W = \{(a, b, c) : a = b = c\}$

Any vector $\in W$ has the form $r\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

↑ basis. $\dim W = 1$.

Question 4. Solve the initial value problem $x' = Ax$ with

$$A = \begin{bmatrix} -2 & -2.5 \\ 10 & -2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

You will find that the eigenvalues and e-vecs of A are $\lambda = -2 \pm 5i$ and $V = \begin{bmatrix} i \\ 2 \end{bmatrix}$

so $x(t) = \begin{bmatrix} i \\ 2 \end{bmatrix} e^{-2t}(\cos 5t + i \sin 5t)$.

The real and imaginary parts form a basis for the solution space. Thus the general soln is

$$x(t) = c_1 \begin{bmatrix} -\sin 5t \\ 2\cos 5t \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} \cos 5t \\ 2\sin 5t \end{bmatrix} e^{-2t}$$

To satisfy $x(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ we need

$$c_1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow c_1 = 1.5, \quad c_2 = 3$$

$$\Rightarrow x(t) = 1.5 \begin{bmatrix} -\sin 5t \\ 2\cos 5t \end{bmatrix} e^{-2t} + 3 \begin{bmatrix} \cos 5t \\ 2\sin 5t \end{bmatrix} e^{-2t}$$

Question 5. Let $V = P(t)$, the vector space of real polynomials. Determine whether or not W is a subspace of V. Explain.

(a) W consists of all polynomials with integer coefficients.

No because multiples of polynomials in W do not always belong to W, e.g.,

$f(t) = 3 + 6t + 7t^2 \in W$

but $\frac{1}{2}f(t) = \frac{3}{2} + 3t + \frac{7}{2}t \notin W$

(b) W consists of all polynomials with degree ≥ 6 and the zero polynomial.

No. $\left.\begin{aligned} f_1(t) &= t^6 \\ f_2(t) &= -t^6 + t^5 \end{aligned}\right\} \in W$

but $f_1(t) - f_2(t) = t^5 \notin W$

(c) W consists of all polynomials with only even powers of t.

Yes (show closure)

assume zero polynomial is one with even power of t.

Question 6. (a) Find the matrix of fundamental solutions for the homogeneous system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$y' = y \Rightarrow y(t) = c_2 e^t$$

$$x' = x + y = x + c_2 e^t \Rightarrow x(t) = c_2 t e^t + c_1 e^t$$

$$M = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

(b) Find the general solution for the nonhomogeneous system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$$

$$M\int M^{-1}\begin{bmatrix} e^t \\ e^{-t} \end{bmatrix} dt = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \int \begin{bmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix} dt$$

$$= \begin{pmatrix} te^t + \frac{1}{4}e^{-t} \\ -\frac{1}{2}e^{-t} \end{pmatrix}$$

$$x(t) = \begin{bmatrix} c_1 e^t + c_2 te^t + te^t + \frac{1}{4}e^{-t} \\ c_2 e^t - \frac{1}{2}e^{-t} \end{bmatrix}$$

Question 7. (a) Find an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}, \quad A = \begin{bmatrix} 6 & -2 \\ 6 & -1 \end{bmatrix}$$

Find eigenvalues ~~of~~ and eigenvectors of A.

$\det(A - \lambda I) = 0 \Rightarrow \lambda = 2, 3.$

$\lambda = 2, \quad v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $\quad \lambda = 3, \quad v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

So $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

(b) Find A^{-1} without directly inverting A.

$A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}$

$P^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$ and $D^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1/6 & 1/3 \\ -1 & 1 \end{bmatrix}$

Question 8. Let S consist of the following vectors in $\Re^4$:

$$u_1 = (1,1,0,-1), \quad u_2 = (1,2,1,3), \quad u_3 = (1,1,-9,2), \quad u_4(16,-13,1,3)$$

Of course you need to do this in a test!

(a) Show that S is orthogonal and a basis of $\Re^4$.

Compute $u_1 \cdot u_2 = 1+2+0-3=0$ ✓

$u_2 \cdot u_3 = \cdots = 0 \quad u_1 \cdot u_3 = \cdots = 0$

$u_2 \cdot u_4 = \cdots = 0 \quad u_1 \cdot u_4 = \cdots = 0$

$u_3 \cdot u_4 = \cdots = 0$

Thus, S is orthogonal, and S is linearly independent. According, S is a basis for $\mathbb{R}^4$ because any four linearly independent vectors form a basis of $\mathbb{R}^4$.

(b) Find the coordinates of an arbitrary vector $v = (a,b,c,d)$ in $\Re^4$ relative to the basis S.

$$c_1 = \frac{v \cdot u_1}{u_1 \cdot u_1} = \frac{a+b-d}{3}$$

$$c_2 = \frac{v \cdot u_2}{u_2 \cdot u_2} = \frac{a+2b+c+3d}{15}$$

$$c_3 = \frac{v \cdot u_3}{u_3 \cdot u_3} = \frac{a+b-9c+2d}{87}$$

$$c_4 = \frac{v \cdot u_4}{u_4 \cdot u_4} = \frac{16a-13b+c+3d}{435}$$

✓