Applied Harmonic Analysis
Math 348

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Course Synopsis
The objective of this course is to get acquainted with important ideas and tools in Harmonic Analysis that significantly impact applications in several fields, in particular signal processing, scientific computation and machine learning. It is directed to graduate students in mathematics, engineering and computer science.
The course will be taught at different levels: the main objective of the course is to offer a panorama of the material rather than the fine details. In many occasions I will be reviewing and summarizing material that could be treated in much greater detail and depth - both from the point of view of the mathematics involved and of the applications. References will be provided and students will volunteer to give presentations on relevant topics or papers of interest, of course with assistance of the instructor during the preparation phase.

Background
We will assume a solid background in linear algebra, in particular vector spaces, normed and inner-product spaces. The book by Halmos is a nice reference for this material. It is desirable, but not required, to know the basics of Hilbert spaces, and of Lebesgue L^p-spaces. Basic probability theory and theory of Markov chains will also be required in the last part of the course. Finite-dimensional spaces, finite probability spaces and Markov chains are enough to understand the material presented in the course - realizing that moving to infinite-dimensional spaces, in a quantitative fashion, is necessary and useful will be a recurring theme. Students seeking more detailed information regarding the above requirements are welcomed to contact the instructor.

Topics
In the first week of the course I will overview the ideas, topics and applications covered during the course, as a motivation for the contents and structure of the course as a whole. We will then review the basics of approximation theory in normed function spaces. Topics will include bases, frames, tight frames; linear and nonlinear approximation.

Fourier Analysis. Basics of Fourier analysis: discrete and continuous Fourier transforms. Windowed Fourier transform. Computational aspects: the Fast Fourier Transform. Approximation properties of Fourier series. Applications: audio and image compression and denoising.

Wavelets and Multiscale Analysis. Basics of wavelet analysis. Multiresolution analysis, filter banks. Computational aspects: the Fast Wavelet Transform. Wavelet frame reconstruction algorithms. Approximation properties of wavelet series. Applications. One dimensional signal processing: audio compression and denoising (revisited). Two-dimensional signal processing: image compression, denoising, wavelet and curvelet methods. Three-dimensional signal processing: video and hyperspectral images, Fourier, wavelet and wavelet packet methods. High-dimensional signal processing: a fast tour through higher-dimensional wavelet constructions.

Fourier and Wavelet methods on graphs and manifolds. Signal processing on manifolds and graphs. Fourier and wavelet analysis can be generalized to rather general families of spaces, such as graphs and manifolds, and these generalizations are finding use in many different applications. This is a very recent and quickly growing area of research, in view of applications to a wide range of problems arising in disparate fields. We will introduce these generalizations of Fourier and wavelet analysis: basics of spectral graph theory, random walks and diffusion processes on graphs, eigenfunctions of the Laplacian and wavelets on graphs. We will discuss applications to:

Machine learning, in particular semi-supervised learning, or supervised learning on graphs and manifolds. A typical question in this context is: suppose you have a large number of documents (say, web pages), and few of them are labeled (by topic; or scored, by level of interest/relevance); how can you label (or score) automatically and accurately all the non-labeled documents? We will interpret this as a problem in approximation/regularization theory and use generalization of Fourier and wavelet analysis to tackle this problem. We will review standard methods in the field, discussing connections, novelties and current research directions.

Reinforcement learning and Markov decision processes. This is a rather general and flexible framework for having agents learning how to optimally interact with their environment in order to maximize a utility function. This can be cast as an approximation-optimization problem, for which generalized Fourier and wavelet techniques are natural and efficient tools. We will review standard methods in the field, discussing connections, novelties and current research directions.

Nonlinear image analysis, segmentation and denoising. Overview of PDE-based methods, vs. the basis methods discussed in the previous part of the course. Novel methods based on high-dimensional image representations.