A proof that a discrete delta function is second order accurate

J. Thomas Beale [a]

[a] *Department of Mathematics, Duke University, Box 90320, Durham, NC 27708-0320*

Abstract

It is proved that a discrete delta function introduced by P. Smereka [10] gives a second-order accurate quadrature rule for surface integrals using values on a regular background grid. The delta function is found using a technique of A. Mayo [7]. It can be expressed naturally using a level set function.

Key words: discrete delta function, level set function, surface integral

There is considerable interest in designing accurate discrete delta functions for surfaces in a domain covered by a rectangular grid. They can provide quadrature rules for surface integrals using values at regular grid points [2,10–12]. Such a rule is especially useful when the surface is represented by a level set function. In [10] P. Smereka constructed a discrete delta function as the truncation error in applying the discrete Laplacian to a "Green's function" for the exact delta function on the surface. To find the truncation error, he used the technique of A. Mayo [7,8] for solving differential equations with interfacial conditions, in which jump conditions are built into the difference operators on a regular grid. (The immersed interface method [3,5], the EJIIM [13,9], and the ghost fluid method [6] are related to Mayo's technique.) Smereka also showed how to express this delta function in terms of a level set function. He conjectured that the resulting quadrature rule for surface integrals is second order accurate and verified the accuracy in numerical examples. In this note we give a simple proof of this fact.

Suppose Γ is a closed curve in $\mathbb{R}^2$ or a closed surface in $\mathbb{R}^3$, bounding a set which is contained in a rectangular domain Ω. The problem is to design a

Email address: beale@math.duke.edu (J. Thomas Beale).
URL: http://math.duke.edu/faculty/beale (J. Thomas Beale).

weight function w^h at grid points on a square grid Ω_h, concentrated near Γ, so that, for any smooth function f defined near the curve Γ in $\mathbb{R}^2$,

$$\int_\Gamma f(x)\,ds(x) \;=\; \sum_{ih\in\Omega_h} f(ih)w^h(ih)\,h^2 \;+\; O(h^2) \tag{1}$$

or near the surface Γ in $\mathbb{R}^3$,

$$\int_\Gamma f(x)\,dS(x) \;=\; \sum_{ih\in\Omega_h} f(ih)w^h(ih)\,h^3 \;+\; O(h^2)\,. \tag{2}$$

Arclength and surface area are special cases. Smereka's w^h has support on the grid points ih within distance h of Γ, i.e., $w^h(ih) = 0$ at other points. We will prove that (2) holds, with w^h as in [10], assuming Γ is a smooth surface in $\mathbb{R}^3$. The case of a curve in $\mathbb{R}^2$ is entirely similar.

Smereka's procedure is as follows: Let δ_Γ be the distribution, or generalized function, restricting to Γ; that is, for smooth f on Ω,

$$\int_\Omega f\delta_\Gamma\,dx \;=\; \int_\Gamma f\,dS\,. \tag{3}$$

Let g be the solution of

$$\Delta g = \delta_\Gamma \;\text{ in }\; \Omega\,, \qquad g = 0 \;\text{ on }\; \partial\Omega\,. \tag{4}$$

Assuming Γ is smooth, g is piecewise smooth, i.e., smooth and harmonic on each region bounded by Γ, with the jump conditions

$$[g] = 0\,, \quad [\partial_n g] = 1 \quad \text{on }\; \Gamma\,. \tag{5}$$

where ∂_n is the normal derivative on Γ. In fact g can be thought of as a single layer potential on Γ. Now let Δ_h be the usual second-order discrete Laplacian on Ω_h, and let τ^h be the truncation error

$$\Delta_h g \;=\; \tau^h \quad \text{on }\; \Omega_h\,. \tag{6}$$

Smereka constructs the weights w^h from expressions for τ^h, using Mayo's technique [7,8]. At a *regular* grid point $ih \in \Omega_h$, for which the stencil of Δ_h does not cross Γ, $\tau^h(ih) = O(h^2)$ as usual. At an *irregular* grid point, τ^h is larger. It can be found to $O(h)$ using the jumps in first and second derivatives of g; see (30) in [10]. These in turn can be expressed in derivatives of the normal and tangent vectors to Γ. (See (41), (47) in [10] for $\mathbb{R}^2$ and Sec. 7.2 for $\mathbb{R}^3$.) Thus τ^h has the form

$$\Delta_h g \;=\; \tau^h \;=\; w^h + O_\Gamma(h) + O(h^2) \quad \text{on }\; \Omega_h\,, \tag{7}$$

where w^h is known analytically, and w^h and $O_\Gamma(h)$ are nonzero only at the irregular points. The errors are uniform. Smereka shows how to write w^h in terms of a level set function; see (45) and Sec. 7 in [10].

To prove that (2) is valid, we may assume f is nonzero only in a neighborhood of Γ, as well as smooth. We begin by writing

$$\int_\Gamma f\,dS \;=\; \int_\Omega f\delta_\Gamma\,dx \;=\; \int_\Omega f\Delta g\,dx \;=\; \int_\Omega g\,\Delta f\,dx\,. \tag{8}$$

(This could be rewritten in an equivalent way using the jump conditions (5) rather than δ_Γ.)

Next we replace the last integral by a sum over grid points. We check that

$$\int_\Omega g\,\Delta f\,dx \;=\; \sum_{ih\in\Omega_h} g(ih)(\Delta f)(ih)\,h^3 \;+\; O(h^2) \tag{9}$$

by comparing the integral over the cell centered at ih with the term in the sum. If the cell intersects Γ, the error in the integrand is $O(h)$, since g is continuous and has bounded derivative. There are $O(h^{-2})$ such cells, contributing a total error of $O(h\cdot h^3\cdot h^{-2}) = O(h^2)$. On each remaining cell the error in the integral is $O(h^2\cdot h^3)$, since g and Δf are C^2. The total error for these cells is $O(h^2\cdot h^3\cdot h^{-3}) = O(h^2)$, and the claim (9) is verified.

We now have

$$\int_\Gamma f\,dS = \sum_{\Omega_h} g\,\Delta f\,h^3 + O(h^2) = \sum_{\Omega_h} g\,\Delta_h f\,h^3 + O(h^2) \tag{10}$$

since $\Delta_h f = \Delta f + O(h^2)$. We can sum by parts and use (7) to obtain

$$\sum_{\Omega_h} g\,\Delta_h f\,h^3 = \sum_{\Omega_h} (\Delta_h g)\,f\,h^3 = \sum_{\Omega_h} \Big(w^h + O_\Gamma(h) + O(h^2)\Big)\,f\,h^3\,. \tag{11}$$

The $O_\Gamma(h)$ error contributes a term of order $h\cdot h^3\cdot h^{-2} = h^2$, and thus is negligible, as is the other error inside. Combining (10) and (11), we arrive at the conclusion (2).

The fact that the integral is accurate to $O(h^2)$ although $\tau^h = O(h)$ on the irregular points is related to a gain in accuracy that has long been noted for solutions of elliptic problems using the methods of [3–5,7,8,13]. Proofs of this phenomenon have been given in [1,4,9] and elsewhere. Closely related to the Green's function g solving (4) is the discrete version g^h which solves

$$\Delta_h g^h \;=\; w^h \;\text{ in }\; \Omega_h\,, \qquad g = 0 \;\text{ on }\; \partial\Omega_h\,. \tag{12}$$

In fact $g^h - g = O(h^2)$ uniformly; this follows from analytical results in [1,9].

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