I will start with a ring we know very well, and talk about its many important generalizations, which we know less well.

Classic question: Given four lines in space, how many others intersect all four?

Um... infinite? Well, put them in general position

Um... zero? Well, look for complex solutions.

Um... still zero? Perhaps at infinity!

Classic answer: Two. As seen on the tetrahedron.

How is this a ring calculation? Let $Gr_2 \mathbb{C}^4 = Gr_1(\mathbb{CP}^3)$ be the space of lines.

Imposing one intersection condition cuts us down to a divisor I will call X_{0101}, and use $[X_{0101}]$.

"General position" means we move that divisor into four transverse positions.

So eventually the calculation is $[X_{0101}]^4 = 2[pt]$ (in Poincaré dual notation) in $H^*(Gr_2\mathbb{C}^4)$

More generally, how to compute in $H^*(Gr_k \mathbb{C}^n)$?

Given $V \in Gr_k \mathbb{C}^n$, look at $(V \cap \mathbb{C}^0) \leq (V \cap \mathbb{C}^1) \leq \dots \leq (V \cap \mathbb{C}^n) = V$

dim: $0 \leq \dots \leq k$

jumps by 0 or 1

Ex $V = \mathbb{C}^k$: 1111 0000000

V generic: 00000 0001111

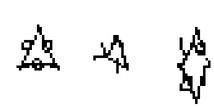
Ex Asking a line in $\mathbb{CP}^3$ to intersect the standard line in a point is asking a plane in $\mathbb{C}^4$ to meet the std plane in dim 1. So 0101

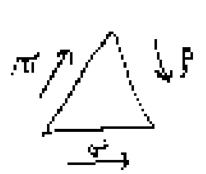
There is actually a Morse stratification by even-dim (complex) cells, so gives basis of H^*.

Thm (KT) The product $[X_\pi][X_\rho] \in H^*(Gr_k \mathbb{C}^n)$ is

$$\sum_{\text{puzzles}} [X_\sigma],$$

where a puzzle is made of puzzle pieces





Why is this positive? Can one set up a bijection?

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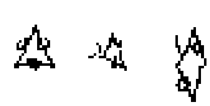

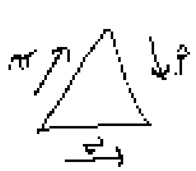

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(3) Equivariant cohomology. The Schubert cycles are T-invt, so give a basis — over $H^*_T(pt) \cong \mathbb{Z}[y_1, y_2, \ldots, y_n]$

(4) Quantum cohomology — counting $\mathbb{CP}^1$'s in Grassmannians.

Even if you don't like these exotic cohomology theories, you can still appreciate the ...

(FF) H^*(Flag mflds).
Given a flag F in $\mathbb{C}^n$, let $R_{ij} = \dim\left(F_i \cap \mathbb{C}^j \,/\, \left((F_{i-1} \cap \mathbb{C}^j) + (F_i \cap \mathbb{C}^{j-1})\right)\right)$.

Then R is a permutation matrix, it turns out, and the stratification by R's is again perfect. And all combinations of these!

What's known?

Q, FF, T, TFF: Abstractly we know these are positive. FF – classical, Q – Witten, Agnihotri, TFF – W. Graham

K: we have a formula! (Buch '00) ... It amounts to allowing [piece] as a new piece.

Very mysterious: the coefficient of $[X_\nu]$ in $[X_\lambda][X_\mu]$ is not positive, but has a predictable sign $(-1)^{\ell(\lambda\mu)-\ell(\nu)}$. Why?

T. Very weird Molev-Sagan ~~rule~~ that doesn't quite do it.

Include a new puzzle piece [piece] and up to rotation

Assign it a "weight" $y_i - y_j$ depending on location in the puzzle.

row shaped

Then coeff of $[X_\nu]$ in $[X_\lambda][X_\mu]$ is $\sum_{\text{puzzles}} \prod_{\text{such pieces } p} wt(\text{piece})$

Ex $[X_{010}][X_{100}] = (y_1 - y_3)[X_{100}]$

$y_2 - y_3$ $\qquad$ $y_1 - y_3$ $\qquad$ $y_1 - y_3$

So this formula is not positive -- but it reduces to the positive formula from before! And is actually the easiest way to prove it, by induction backward from the highest-degree case.