Lefschetz theorems for degeneracy loci

1. Degeneracy loci

Let X be a complex projective variety of dimension n.
E, F vector bundles on X of respective ranks e and f
$u : E \to F$

We are interested in the topology of the loci
$$D_r = \{ x \in X \mid \operatorname{rank}(u_x) \leq r \} .$$

The simplest case is when $E \simeq \mathcal{O}_X$ and F is a line bundle: there is only one locus $D_0 =$ zero set of u.
If F is ample, we have:

Bertini: D_0 is connected if $n-1 > 0$

Lefschetz: If $X - D_0$ is smooth,
$$H^p(X, D_0; \mathbb{Z}) = 0 \quad \text{for } p < n$$

Grothendieck: If X smooth
$$\operatorname{Pic}(X) \xrightarrow{\sim} \operatorname{Pic}(D_0) \quad \text{for } n-1 \geq 3$$

The proof of Lefschetz' theorem is easy: $X - D_0$ is affine, hence has the homology of a CW complex of (real) dimension n, hence $H_q(X - D_0; \mathbb{Z}) = 0$ for $q > n$. By Lefschetz duality, if $X - D_0$ is smooth, $H^p(X, D_0; \mathbb{Z}) \simeq H_{2n-p}(X - D_0; \mathbb{Z}) = 0$ for $p < n$.

Example: C smooth projective curve of genus g.
$J^d(C)$ Jacobian of degree d line bundles

$$W^s_d(C) = \{ L \in J^d(C) \mid h^0(C,L) > s \}$$

is a degeneracy loci: let D be a reduced divisor on C of high degree δ. There is an exact sequence

$$0 \to H^0(L) \to \underset{\substack{\text{dimension} \\ \delta + d + 1 - g}}{H^0(L(D))} \xrightarrow{u} \underset{\substack{\text{dimension} \\ \delta}}{H^0(L(D)|_D)}$$

As L varies in $J^d(C)$, there are fibers E_L and F_L of vector bundles and

$$W^s_d(C) = D_{\delta + d + 1 - g - (s+1)}$$

The natural hypothesis needed in the general case to obtain theorems of Bertini, Lefschetz or Grothendieck type is

$$\mathrm{Hom}(E,F) = E^* \otimes F \qquad \underline{\text{ample}}$$

(holds for example above).

A vector bundle G is ample if $\mathcal{O}_G(1)$ is ample on $\mathbb{P}G^*$.
This is used as follows for E trivial

$\mathbb{P}F^* \supset Z = \{ [\ell] \in F^*_x \mid \ell(u(x)) = 0 \}$ is a section of $\mathcal{O}_G(1)$

hence $\mathbb{P}F^* - Z$ is affine.

The projection $\mathbb{P}F^* - Z \to X - D_0$ is a $\mathbb{C}^{f-1}$-bundle hence

$$H^p(X, D_0; \mathbb{Z}) \underset{\substack{\text{Lefschetz} \\ \text{duality}}}{\xrightarrow{\sim}} H_{2n-p}(X - D_0; \mathbb{Z}) \simeq H_{2n-p}(\mathbb{P}F^* - Z; \mathbb{Z})$$
$$= 0 \text{ for } 2n - p > n + f - 1$$
$$\text{i.e. } p \leq n - f$$

In general, the construction (due to Fulton and Lazarsfeld) is more involved. Start with

$$\begin{array}{ccc} \pi: G = G(e-r, E) & \longrightarrow & X \\ \cup & & \cup \\ Y = \{ \Lambda_x \subset E_x \mid \Lambda_x \subset \mathrm{Ker}(u_x) \} & \to & D_r \end{array}$$

Y is the zero locus of

$$v:\ S \subset \pi^* E \xrightarrow{\pi^* u} \pi^* F$$

where S is the tautological bundle.

The <u>expected dimension</u> of D_r is

$$\delta(r) \quad \dim G - (e-r)f = n - (e-r)(f-r)$$

But $\pi^* F \otimes S^*$ is not ample! However, the same vanishing still holds, i.e.

$$H^p(G, Y; \mathbb{Z}) = 0 \quad \text{for} \quad p \leq \delta(r).$$

when $X \setminus D_0$ is smooth

This implies: (a slightly different argument works when X is singular)

<u>Bertini</u> D_r is non-empty for $\delta(r) \geq 0$

connected for $\delta(r) > 0$.

<u>Lefschetz</u> <u>when</u> $D_{r-1} = \emptyset$. In this case, $Y \simeq D_r$ and we get for example

$$H^2(D_r, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}) \oplus \mathbb{Z}\, c_1(\mathrm{Ker}(u))$$

rank r vector bundle on D_r.

for $\delta(r) \geq 3$, $0 < r < e$.

<u>When</u> $D_{r-1} \neq \emptyset$, we can still compare the Leray spectral sequences for π and π' ($\pi'_* \mathbb{Z} = \mathbb{Z}_{D_r}$, $R^2\pi'_* \mathbb{Z} = \mathbb{Z}_{D_{r-1}}$)

$$\begin{tikzcd}
0 \arrow[r] & H^2(X) = E_3^{20} \arrow[r] \arrow[d] & H^2(G) = E_\infty^{2} \arrow[r] \arrow[d] & H^0(X) = E_3^{02} \arrow[r, "0", "d_3^{02}"'] \arrow[d] & H^3(X) = E_3^{30} \arrow[d] \\
0 \arrow[r] & H^2(D_r) \arrow[r] & H^2(Y) \arrow[r] & H^0(D_{r-1}) \arrow[r] & H^3(D_r)
\end{tikzcd}$$

injective for $\delta(r) \geq 2$ — bijective for $\delta(r) \geq 3$ — injective if $D_{r-1} \neq \emptyset$

We get $\quad H^2(X) \xrightarrow{\sim} H^2(D_r) \quad$ for $\delta(r) \geq 3$ and $\delta(r-1) \geq 1$.

The same techniques yield

<u>Theorem</u> : X smooth projective complex variety, $E^* \otimes F$ ample

If $[m/2] \le r$ and $\delta(r - [m/2]) \geq \varepsilon(m)$

$$H^p(X, \mathbb{Z}) \to H^p(D_r, \mathbb{Z}) \quad \text{is bijective for } p \le m$$

where

$m =$	0	1	2	3	4	5	....
$\varepsilon(m) =$	1	2	0	1	0	1	....

Grothendieck's method of proof for Picard groups is based on vanishing theorems (which may fail for vector bundles) and assumes that D_r has the expected dimension (too restrictive).

Another method is to try to relate $H^p(D, \mathbb{C})$ to $H^p(D, \mathcal{O}_D)$ and use exponential sequence.

If D_r is smooth (this is rarely the case!)

$$H^p(D_r, \mathcal{O}_{D_r}) \simeq Gr_F^0 H^p(D_r, \mathbb{C})$$

This is not true in general: $H^p(D_r, \mathbb{C})$ carries a mixed Hodge structure, but some information is still available.

For example $\quad H^1(D_r, \mathcal{O}_{D_r}) = Gr_F^0 Gr_1^W H^1(D, \mathbb{C})$

when D_r is <u>normal</u>.

Using these ideas, we get

<u>Theorem</u> $\quad X$ smooth projective complex variety, $E^* \otimes F$ ample

D_r normal

$$\mathrm{Pic}(D_r) \simeq \mathrm{Pic}(X) \oplus \mathbb{Z}[\det(\mathrm{Ker}\, u)] \quad \text{if } D_{r-1} = \phi \text{ and } 0 < r < e$$

$$\simeq \mathrm{Pic}(X) \quad \text{otherwise.}$$

<u>Example</u> : C smooth projective curve

$W_d(C) = W_d^0(C)$ has the expected dimension d and is normal.

If C has a g^1_d, $\operatorname{Pic}(JC) \cong \operatorname{Pic}(W_d(C))$ for $d \geq 3$.
Note that $C^{(d)} \longrightarrow W_d(C)$, the Abel-Jacobi map, is a desingularization. (the smooth locus of $W_d(C)$ is $W_d(C) - W^1_d(C)$). It follows that the cokernel of

$$\operatorname{Pic}(W_d(C)) \hookrightarrow \operatorname{Pic}(W_d(C)_{\text{smooth}})$$

is $\mathbb{Z}$ when C is non hyperelliptic
$\mathbb{Z}/(g-d+1)\mathbb{Z}$ otherwise.

2. Alternate degeneracy loci

This is the case where we deal with an antisymmetric $u: E \to E^*$. One uses a similar method (Pragacz, Tu): look again at

$$\pi: G = G(e-r, E) \longrightarrow X$$

The loci we are interested in are

$$A_r = \{ x \in X \mid \operatorname{rank}(u_x) \leq 2r \}$$

An antisymmetric form has rank $\leq 2r$ iff it has an isotropic $(e-r)$-plane. Set

$$Y = \text{zero locus of } (\Lambda^2 E^* \to \Lambda^2 S^*) \quad ;$$

it follows that $\pi(Y) = A_r$

The expected dimension is $\alpha(r) = n - \binom{e-2r}{2}$

The natural assumption is $\Lambda^2 E^*$ ample

The Bertini theorem was proved by Tu: A_r is non-empty for $\alpha(r) \geq 0$, connected for $\alpha(r) > 0$.

Using the same methods as above, one gets a Lefschetz theorem. The results are different. I'll explain why for the H^2. Assume $A_{r-1} = \phi$. In the standard case, $H^2(D_r, \mathbb{Z})$ had one extra factor $\mathbb{Z}$ generated by $c_1(\mathrm{Ker}(u))$. Here,

$$0 \to \mathrm{Ker}(u) \to E \xrightarrow{u} E^* \to \mathrm{Ker}(u)^* \to 0$$

implies $\quad 2\, c_1(\mathrm{Ker}(u)) = 2\, c_1(E) \in H^2 \quad (X)$.

One can actually divide by 2. We get for example

<u>Theorem</u> X smooth projective complex variety, $\Lambda^2 E^*$ ample.

1) If $[\frac{m}{4}] \leq r$ and $\alpha(r - [m/4]) \geq \varepsilon'(m)$

$H^p(X, \mathbb{Z}) \to H^p(\quad A_r; \mathbb{Z})$ bijective for $p \leq m$

$m = 0\,1\,2\,3\,4\,5\,6\,7\,8$
$\varepsilon'(m) = 1\,2\,3\,4\,0\,1\,2\,3\,0\,1\,2\,3\,..$

2) If A_r normal, $\alpha(r) \geq 3$ $\qquad \mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(A_r)$

3. <u>Orthogonal degeneracy loci</u>

The following situation often occurs:

V vector bundle of rank $2m$ with a non-degenerate quadratic form.

E, F maximal isotropic subbundles.

$$O^r = \{ x \in X \mid \dim(E_x \cap F_x) \geq r \text{ and } \equiv r \bmod 2 \}$$

<u>Notes</u> 1) The parity of $\dim(E_x \cap F_x)$ is locally constant, so if X is connected, either $X = O^0$ and $O^{2r+1} = \phi$

or $X = O^1$ and $O^{2r} = \phi$.

2) If $u: E \to E^*$ is antisymmetric set

$V = E \oplus E^*$ with the hyperbolic form

$E \oplus \{0\}$ and $\mathrm{Im}(\mathrm{Id}_E, u)$ are totally isotropic

and $\quad O^{e-2r} = A_r$

However, the natural hypothesis here is $E^* \otimes F^*$ ample.

Examples 1) $\pi : \tilde{C} \to C$ connected double étale cover of smooth curves, genus $2g-1$ and g.
$$P = \{ [L] \in J^{2g-2} \tilde{C} \mid L \otimes \sigma^* L \simeq \omega_{\tilde{C}} \quad h^0(\tilde{C}, L) \text{ even} \}$$
is a principally polarized abelian variety.
Mumford showed that
$$W^{2s} = \{ [L] \in P \mid h^0(\tilde{C}, L) \geq 2s \}$$
is an orthogonal degeneracy locus (W^2 is a theta divisor) ; in this set-up, $E^* \otimes F^*$ is ample (they are the restriction to P of the corresponding vector bundles on $J^{2g-2}\tilde{C}$ used to define $W^{2s}_{2g-2}(\tilde{C})$).

2) $\mathcal{C} \to T$ family of smooth projective curves with a theta characteristic (i.e. a line bundle $\mathcal{L}$ on $\mathcal{C}$ s.t. $\mathcal{L}^{\otimes 2} \simeq \omega_{\mathcal{C}/T}$).
$$\{ t \in T \mid h^0(\mathcal{C}_t, \mathcal{L}_t) \geq r \text{ and } \equiv r \bmod 2 \}$$
is an orthogonal degeneracy locus.

Bertini If the expected dimension $n - \binom{r}{2}$ is ≥ 0, O^r is non-empty (Bertram) : If it is > 0, O^r is connected.

.. I don't know how to prove a Lefschetz theorem in this case. The problem is that there is no known "resolution" of O_r by the zero locus of a section of a vector bundle. This is the main reason for example why the class of O_r has only been recently computed (Fulton–Pragacz), whereas the Giambelli–Thom–Porteous formula for the usual degeneracy loci is much older