Third Midterm Examination,
MAT 108, Differential Equations.

Last Name

First Name

Student ID Number

Remarks:

- The test contains 5 pages, including the cover sheet.
- Do *not de-staple* the test.
- The use of books, notes, and calculators is not allowed.
- You can use scratch paper during the exam. However, you *cannot submit any work on your scratch paper* with the exam.
- Use both sides of the sheets.

Exercise I. Let f be a function such that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right) + b_n \sin\left(\frac{n\pi}{p}x\right),$$

with a_0, a_i and b_i, $i = 1, \cdots, n$, and p are real numbers.

- Prove the Parseval's identity

$$\frac{1}{2p}\int_{-p}^{p} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2). \tag{1}$$

- Show that the Fourier series decomposition of the 2π-periodic function $f(x) = x/2$ for $-\pi < x < \pi$ is given by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx), \quad -\pi < x < \pi. \tag{2}$$

- Use Parseval's identity to deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{24}$ and $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$.

Exercise II. Find the Laplace transform of the following functions

- $f(t) = 1$ if $0 \leq t \leq \pi$, $f(t) = -1$ if $\pi \leq t \leq 2\pi$ and $f(t + 2\pi) = f(t)$.
- $f(t) = t^2 e^{-t} \cos(2t)$.
- $f(t) = \delta(t - \pi/6) \sin(t) \cos(2t)$.

Exercise III. Consider the following eigenvalue problem

$$x^2y'' - xy' + (\lambda - 2)y = 0, \tag{3}$$

with $y'(1) = 0$, $y(L) = 0$, $L > 1$. Find its solution if it exists.

Exercise IV. Consider the following differential equation

$$y'' + y = F(t), \tag{4}$$

with $y(0) = 0$, $y'(0) = 0$.

- Using the Laplace transform, find the general solution of (4).
- Using the preceding result, find the solution when $F(t) = \sin(\omega t)$ with ω a real number [Hint: distinguish between the cases $\omega = 1$ and $\omega \neq 1$] [note: the question will not be graded if you use a different method].