1. Topological spaces

Definition 1.1. We say a family of sets $\mathcal{T}$ is a **topology** if

(i) $\cup\mathcal{U} \in \mathcal{T}$ whenever $\mathcal{U} \subset \mathcal{T}$;
(ii) $\cap\mathcal{F} \in \mathcal{T}$ whenever $\mathcal{F} \subset \mathcal{T}$ and $\mathcal{F}$ is finite and nonempty.

Note that if $\mathcal{T}$ is a topology then $\emptyset = \cup\emptyset \in \mathcal{T}$ since $\emptyset \subset \mathcal{T}$.

Definition 1.2. Let X be a set. A family $\mathcal{T}$ of sets is a **topology for** X if $\mathcal{T}$ is a topology and $X = \cup\mathcal{T}$.

Remark 1.1. Suppose $\mathcal{T}$ is a topology for the set X. Since $\mathcal{T} \subset \mathcal{T}$ and $X = \cup\mathcal{T}$ we have $X \in \mathcal{T}$. Moreover, if $U \in \mathcal{T}$ then $U \subset \cup\mathcal{T} \subset X$.

Definition 1.3. A **topological space** is an ordered pair $(X, \mathcal{T})$ such that X is a set and $\mathcal{T}$ is a topology for X; in this context the members of $\mathcal{T}$ are called **open sets** and a subset F of X such that $X \sim F$ is open is called **closed**.

It follows directly from the DeMorgan laws that the intersection of a nonempty family of closed sets is closed and that the union of a finite family of closed sets is closed.

Note that ϕ and X are always open and closed.

One often says "X is a topological space" so mean that there is $\mathcal{T}$ such that $(X, \mathcal{T})$ is a topological space.

Definition 1.4. Whenever $a \in \mathbb{R}n$ and r is a positive real number we let

$$\mathbf{U}^a(r) = \{x \in \mathbb{R}n : |x - a| < r\} \quad \text{and} \quad \mathbf{B}^a(r) = \{x \in \mathbb{R}n : |x - a| \le r\}$$

and call these sets the **open ball with center a and radius r** and the **closed ball with center a and radius r**, respectively. We say a subset U of $\mathbb{R}n$ is **open** if for each $a \in U$ there is $\epsilon > 0$ such that

$$\mathbf{U}^a(\epsilon) \subset U. \tag{1}$$

Theorem 1.1. The family of open sets is a topology for $\mathbb{R}n$.

Proof. Suppose $a \in \mathbb{R}n$; then for *any* $\epsilon > 0$ we have $\mathbf{U}^a(\epsilon) \subset \mathbb{R}n$ so $\mathbb{R}n$ is open. Thus the union of the family of open sets is $\mathbb{R}n$.

Suppose $\mathcal{U}$ is a family of open subsets of $\mathbb{R}n$ and $a \in \bigcup\mathcal{U}$; then for some $U \in \mathcal{U}$ we have $a \in U$. Since U is open there is $\epsilon > 0$ such that $\mathbf{U}^a(\epsilon) \subset U$; since $U \subset \bigcup\mathcal{U}$ we infer that $\mathbf{U}^a(\epsilon) \subset \bigcup\mathcal{U}$. Thus $\bigcup\mathcal{U}$ is open.

Suppose $\mathcal{F}$ is a finite family of open subsets of $\mathbb{R}n$ and $a \in \bigcap\mathcal{F}$. For each $U \in \mathcal{F}$ let $\rho(U) = \sup\{\epsilon : \mathbf{U}^a(\epsilon) \subset U\}$ and note that $0 < \rho(U) \le \infty$ since U is open. Let $\sigma = \min\{\rho(U) : U \in \mathcal{F}\}$. Since $\mathcal{F}$ is finite we have $0 < \sigma \le \infty$. Choose ϵ such that $0 < \epsilon < \sigma$. Then

$$\mathbf{U}^a(\epsilon) \subset U \quad \text{whenever } U \in \mathcal{F}$$

so that $\mathbf{U}^a(\epsilon) \subset \bigcap\mathcal{F}$. Thus $\bigcap\mathcal{F}$ is open. □

Exercise 1.1. Show that $\mathbf{U}^a(r)$ is open and $\mathbf{B}^a(r)$ is closed whenever $a \in \mathbb{R}n$ and r is a positive real number.

1.1. Let us fix a topological space X.

Definition 1.5. Suppose $A \subset X$. We let the **interior of** A be the set of those points a such that $a \in U \subset A$ for some open set U. We let the **closure of** A be the set of those points a such that $A \cap U \neq \emptyset$ whenever U is open and $a \in U$.

We will use the abbreviations

$$\textbf{int}\, A, \quad \textbf{cl}\, A$$

for the interior of A and the closure of A, respectively

Theorem 1.2. Suppose $A \subset X$. We have

$$\textbf{int}\, A \subset A \quad \text{and} \quad A \subset \textbf{cl}\, A.$$

Proof. This follows directly from the definitions. □

.

Theorem 1.3. Suppose $A \subset X$. We have

$$X \sim \textbf{cl}\, A = \textbf{int}\,(X \sim A) \quad \text{and} \quad X \sim \textbf{int}\, A = \textbf{cl}\,(X \sim A).$$

Proof. Suppose $a \in X$.

We have $a \in X \sim \textbf{cl}\, A$ iff there is an open set U such that $a \in U$ and $A \cap U = \emptyset$ iff there is an open set U such that $a \in U$ and $U \subset X \sim A$ iff $a \in \textbf{int}\,(X \sim A)$.

We have $a \in X \sim \textbf{int}\, A$ iff $U \not\subset A$ whenever U is an open set and $a \in U$ iff $U \cap (X \sim A) \neq \emptyset$ whenever U is an open set iff $x \in \textbf{cl}\,(X \sim A)$. □

Corollary 1.1. Suppose $A \subset X$. Then

$$\textbf{int}\, A = X \sim \textbf{cl}\,(X \sim A) \quad \text{and} \quad \textbf{cl}\, A = X \sim \textbf{int}\,(X \sim A).$$

Proof. Replace A by $X \sim A$ in the preceding Theorem. □

Definition 1.6. Suppose $A \subset X$. We let the **boundary of** A be the set of those points a such that if $A \cap U \neq \emptyset$ and $U \sim A \neq \emptyset$ whenever U is open and $a \in U$. We will use the abbreviation

$$\textbf{bdry}\, A$$

for the boundary of A

Theorem 1.4. Suppose $A \subset X$. Then

$$\textbf{bdry}\, A = \textbf{cl}\, A \cap \textbf{cl}\,(X \sim A).$$

Proof. This is an immediate consequence of the definition of boundary. □

Corollary 1.2. Suppose $A \subset X$. Then

$$\textbf{bdry}\, A = \textbf{bdry}\,(X \sim A).$$

Proof. This follows directly from the previous Theorem since $X \sim (X \sim A) = A$. □

Theorem 1.5. Suppose $A \subset X$. Then

$$\textbf{int}\, A = \bigcup \{U : U \text{ is an open set and } U \subset A\}$$

and

$$\textbf{cl}\, A = \bigcap \{F : F \text{ is a closed set and } A \subset F\}.$$

Proof. The first of these statements is a direct consequence of the definition of $\mathbf{int}\, A$. To prove the second, we apply the result just proved to $\mathbf{int}\,(X \sim A)$ to obtain

$$\begin{aligned} X \sim \mathbf{cl}\, A &= \mathbf{int}\,(X \sim A) \\ &= \bigcup\{U : U \text{ is an open set and } U \subset X \sim A\} \\ &= \bigcup\{U : U \text{ is an open set and } A \subset X \sim U\} \\ &= X \sim \bigcap\{X \sim U \text{ is an open set and } A \subset X \sim U\} \\ &= X \sim \bigcap\{F : F \text{ is a closed set and } A \subset F\}. \end{aligned}$$

□

Corollary 1.3. Suppose $A \subset X$. Then $\mathbf{int}\, A$ is open, $\mathbf{cl}\, A$ is closed and $\mathbf{bdry}\, A$ is closed.

Proof. That the interior of A is open is an immediate consequence of the definition of open set. That the closure of A is closed follows from the fact that the intersection of a nonempty family of closed sets is closed as does the fact that the boundary of A is closed. □

Theorem 1.6. Suppose A is a subset of X. Then

$$\{\mathbf{int}\, A, \mathbf{bdry}\, A, \mathbf{int}\,(X \sim A)\}$$

is a partition of X and

$$\{\mathbf{int}\, A, \mathbf{bdry}\, A\}$$

is a partition of $\mathbf{cl}\, A$.

Proof. Let $\mathcal{A} = \{\mathbf{int}\, A, \mathbf{int}\,(X \sim A)\}$. Since the interiors of A and $X \sim A$ are subsets of A and $X \sim A$, respectively, we find that $\mathcal{A}$ is disjointed. Moreover,

$$X \sim \bigcup \mathcal{A} = (X \sim \mathbf{int}\, A) \cap (X \sim \mathbf{int}\,(X \sim A)) = \mathbf{cl}\,(X \sim A) \cap \mathbf{cl}\, A = \mathbf{bdry}\, A.$$

Thus the first assertion is proved.

To prove the second, we have only to note that $X \sim \mathbf{int}\,(X \sim A) = \mathbf{cl}\, A$. □

Definition 1.7. Suppose A is a subset of X and $a \in X$. We say A is a **neighborhood of** a if a is an interior point of A. We say a is an **isolated point** of A if

$$A \cap U = \{a\}$$

for some open set U. We say a is an **accumulation point for** A if

$$A \cap (U \sim \{a\}) \neq \emptyset$$

for each open subset U of X such that $a \in U$. We let

$$\mathbf{iso}\, A = \{a \in X : a \text{ is an isolated point of } A\}$$

and we let

$$\mathbf{acc}\, A = \{a \in X : a \text{ is an accumulation point for } A\}.$$

Theorem 1.7. Suppose A is a nonempty subset of $\mathbb{R}$. If A has an upper bound then $\sup A \in \mathbf{cl}\, A$ and if A has a lower bound then $\inf A \in \mathbf{cl}\, A$.

Proof. Suppose A has an upper bound. Then $-\infty < \sup A < \infty$. Let U be an open subset of $\mathbb{R}$ such that $\sup A \in U$. We need to show that $U \cap A \neq \emptyset$. Since U is open there is $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subset U$. By a previous Theorem there is $a \in A$ such that $\sup A < a + \epsilon$. Thus, as $a \leq \sup A$ we have

$$a \in (\sup A - \epsilon, \sup A) \subset (\sup A - \epsilon, \sup A + \epsilon) \subset U$$

so that $a \in U$ and, therefore, $U \cap A \neq \emptyset$, as desired.

We leave the proof of the second assertion of the Theorem to the reader. □

Theorem 1.8. Suppose A is a subset of X. Then $\mathbf{acc}\, A$ is closed and

$$\{\mathbf{iso}\, A, \mathbf{acc}\, A\}$$

is a partition of $\mathbf{cl}\, A$.

Proof. This is a direct consequence of the definitions. □

1.2. Relative topologies. We suppose throughout this subsections that $(X, \mathcal{T})$ is a topological space and $A \subset X$.

Definition 1.8. We let

$$\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}.$$

One easily verifies that $\mathcal{T}_A$ is a topology for A which we call the **relative topology for** A. We say a subset B of A is **open relative to** A if $B \in \mathcal{T}_A$. We say a subset B of A is **closed relative to** A if $A \sim B$ is relatively open.

Proposition 1.1. Suppose $B \subset A$. Then B is open relative to A if and only $B = A \cap U$ for some open subset U of X and B is closed relative to A if and only if $B = A \cap F$ for some closed subset F of X.

Proof. The first of these assertions is just a repetition of the definition of relative openness.

Suppose B is closed relative to A. Then $A \sim B$ is open relative to A so there is an open subset U of X such that $A \sim B = A \cap U$. Now

$$B = A \sim (A \sim B) = A \sim (A \cap U) = A \cap (X \sim U)$$

so if $F = X \sim U$ then F is a closed subset of X and $B = A \cap F$.

Suppose F is a closed subset of X and $B = X \cap F$. Then

$$A \sim B = A \sim (X \cap F) = X \cap (X \sim F$$

so, as $X \sim F$ is open, $A \sim B$ is relatively open. □

1.3. Connectedness. Suppose X is topological space.

Theorem 1.9. Suppose $A \subset X$. The following are equivalent.

(i) If E and F are open subsets of X, $A \cap E \cap F$ is empty and $A \subset E \cup F$ then either $A \subset E$ or $A \subset F$.

(ii) If G and H are relatively open subsets of A, $G \cap H = \emptyset$ and $A = G \cup H$ then either $A \subset G$ or $A \subset H$.

(iii) If E and F are closed subsets of X, $A \cap E \cap F$ is empty and $A \subset E \cup F$ then either $A \subset E$ or $A \subset F$.

(iv) If G and H are relatively closed subsets of A, $G \cap H = \emptyset$ and $A = G \cup H$ then either $A \subset G$ or $A \subset H$.

Proof. That (i) is equivalent to (ii) is an immediate consequence of the definition of relatively open subsets of A. That (iii) is equivalent to (iv) is an immediate consequence of the characterization of relatively closed sets given in Proposition 1.1. Finally, by considering complements relative to A one immediately infers that G, H satisfy (ii) if and only if G, H satisfy (iii). □

Definition 1.9. We say the subset A of X is **connected** if any of the equivalent conditions in the previous Theorem hold.

Proposition 1.2. The subset A is connected if and only if A is connected with respective to the relative topology for A.

Proof. This follows directly from Theorem 1.9. □

Theorem 1.10. Suppose A is a connected subset of X and

$$A \subset B \subset \mathbf{cl}\, A.$$

Then B is connected.

Proof. Suppose E and F are closed sets, $B \cap E \cap F$ is empty and $B \subset E \cup F$. Since A is connected and $A \subset B$, either $A \subset E$ in which case $\mathbf{cl}\, A \subset E$ because E is closed or $A \subset F$ in which case $\mathbf{cl}\, A \subset F$ because F is closed. Thus B is connected. □

Theorem 1.11. Suppose $\mathcal{A}$ is a nonempty family of connected subsets of X and $\cap\mathcal{A} \neq \emptyset$. Then $\cup\mathcal{A}$ is connected.

Proof. Suppose E and F are open sets, $(\cup\mathcal{A}) \cap E \cap F$ is empty and $\cup\mathcal{A} \subset E \cup F$. Choose a member a of $\cap\mathcal{A}$. Then either (i) $a \in E$ and $a \notin F$ or (ii) $a \notin E$ and $a \in F$.

Suppose (i) holds. Let A be a member of $\mathcal{A}$. As A is connected, either $A \subset E$ or or $A \subset F$. Since $a \in A$ we must have $A \subset E$. Thus $\cup\mathcal{A} \subset E$.

In a similar fashion one shows that $\cup\mathcal{A} \subset F$ if (ii) holds.

Thus $\cup\mathcal{A}$ is connected. □

Definition 1.10. Suppose A is a subset of X and $a \in A$. We let

$$\mathbf{cmp}\,(A, a) = \bigcup\{C : C \text{ is a connected subset of } A \text{ and } a \in C\}.$$

We call this set the **connected component of** a in A. Obviously, if C is a connected subset of A and $a \in C$ then

$$C \subset \mathbf{cmp}\,(A, a).$$

Theorem 1.12. Suppose A is a subset of X. For any $a \in A$ we have

(i) $a \in \mathbf{cmp}\,(A, a)$;
(ii) $\mathbf{cmp}\,(A, a)$ is a connected subset of X;
(iii) $\mathbf{cmp}\,(A, a) = A \cap \mathbf{cl}\,(\mathbf{cmp}\,(A, a))$.

Moreover, $\{\mathbf{cmp}\,(A, a) : a \in A\}$ is a partition of A.

Proof. Suppose $a \in A$.

It follows directly from the definition that $\{a\}$ is connected so that $a \in \mathbf{cmp}\,(A, a)$ so (i) holds. Statement (ii) follows from the preceding Theorem. By a straightforward argument which we leave to the reader one may use statements (i) and (ii) as well as a previous Theorem to infer that $\{\mathbf{cmp}\,(A, a) : a \in A\}$ is a partition of A.

It is trivial that $\mathbf{cmp}\,(A,a)$ is a subset of $A \cap \mathbf{cl}\,(\mathbf{cmp}\,(A,a))$. Suppose $b \in A \cap \mathbf{cl}\,(\mathbf{cmp}\,(A,a))$. Then statement (ii) and the preceding Theorem imply that $\mathbf{cmp}\,(A,a) \cup \{b\}$ is connected. Since this set contains $\{a\}$ by (i) it follows that it is a subset of $\mathbf{cmp}\,(A,a)$ so $b \in \mathbf{cmp}\,(A,a)$. □

Theorem 1.13. Suppose A is a subset of $\mathbb{R}$. Then A is connected if and only if A is an interval.

Proof. Suppose A is connected. Were A not an interval there would be $x, z \in A$ and $y \in \mathbb{R} \sim A$ such that $x < y < z$. Let $E = (-\infty, y)$ and let $F = (y, \infty)$. Then E and F are open, $A \cap E \cap F$ is empty and $A \subset E \cup F$ but $A \not\subset E$ and $A \not\subset F$ so A would not be connected.

On the other hand, suppose A is an interval but that, contrary to the Theorem, A is not connected. Then there would be open sets E and F such that $A \subset E \cup F$ and $A \cap E \cap F = \emptyset$ as well as points x in $A \cap E$ and z in $A \cap F$. Since $A \cap E \cap F$ is empty we have $x \neq z$. Thus

$$x < z \quad \textit{or} \quad z < x.$$

Suppose $x < z$. Let

$$T = \{t : t \in A \cap E \text{ and } t < z\}.$$

Then $x \in T$ so $T \neq \emptyset$ and z is an upper bound for T; letting $y = \sup T$ we find that $x \leq y \leq z$. Since A is an interval we have $y \in A$. Thus *either* (i) $y \in E$ *or* (ii) $y \in F$.

Suppose (i) holds. Since E is open there is $\epsilon > 0$ such that $(y - \epsilon, y + \epsilon) \subset E$. Let $w = \min\{z, y + \epsilon\}$; since $z \in F$ we have $y < z$ so $y < w \leq z$. Since A is an interval we have $[y, w) \subset A$; but this implies $[y, w) \subset T$ which is impossible.

Suppose (ii) holds. Since F is open there is $\eta > 0$ such that $(y - \eta, y + \eta) \subset F$. Let $w = \max\{x, y - \eta\}$; since $x \in E$ we have $x < y$ so $x \leq w < y$. Since A is an interval we have $(w, x] \subset A$; but this implies $(w, x] \subset T$ which is impossible.

In a similar fashion one handles the case $x > z$. □

1.4. Compactness.

Definition 1.11. Suppose $K \subset X$. We say K is **compact** if whenever $\mathcal{U}$ is a family of open subsets of X such that

$$K \subset \bigcup \mathcal{U}$$

there is a finite subfamily $\mathcal{F}$ of $\mathcal{U}$ such that

$$K \subset \bigcup \mathcal{F}.$$

A family $\mathcal{U}$ as above is called and **open covering of** K.

Proposition 1.3. Suppose $K \subset X$. Then K is compact if and only if K is compact with respect to the relative topology for K.

Proof. Suppose K is compact and $\mathcal{V}$ is a family of relatively open subsets of K such that $K = \cup\mathcal{V}$. Let

$$\mathcal{U} = \{U : U \text{ is an open subset of } X \text{ and } K \cap U \in \mathcal{V}\}.$$

If $V \in \mathcal{V}$ then $V = A \cap U$ for some open set U; this implies $\cup\mathcal{V} \subset \mathcal{U}$ so $K \subset \mathcal{U}$. Since I is compact there is a finite subfamily $\mathcal{F}$ of $\mathcal{U}$ such that $K \subset \mathcal{F}$. Let

$$\mathcal{G} = \{K \cap U : U \in \mathcal{F}\}.$$

Then $\mathcal{G}$ is a finite subfamily of $\mathcal{V}$ and $K = \cup\mathcal{G}$ so K is relatively compact.

Suppose K is relatively compact and $\mathcal{U}$ is an open covering of K. Let

$$\mathcal{V} = \{K \cap U : U \in \mathcal{U}\}.$$

The $\mathcal{V}$ is a family of relatively open sets whose union equals K so, as K is relatively compact, there is a finite subfamily $\mathcal{G}$ of $\mathcal{V}$ such that $K = \cup\mathcal{G}$. For each $G \in \mathcal{G}$ let

$$\mathbf{u}(G) = \{U : U \text{ is open and } G = K \cap U\}$$

and note that $\mathbf{u}(G)$ is nonempty. Let c be a choice function for $\{\mathbf{u}(G) : G \in \mathcal{G}\}$ and let $\mathcal{F}$ be the range of c. Then $\mathcal{F}$ is a finite subfamily of $\mathcal{U}$ and $K \subset \cup\mathcal{F}$ so K is compact. □

Definition 1.12. We say X **Hausdorff** if whenever a and b are distinct points of X there are open sets U and V such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

Theorem 1.14. Suppose X is Hausdorff and K is a compact subset of X. Then K is closed.

Proof. We may assume K is nonempty. Suppose $y \in X \sim K$. Let $\mathcal{U}$ be the family of those open sets U corresponding to which there is an open subset V of X such that $y \in V$ and $U \cap V = \emptyset$. Our hypothesis that X is Hausdorff directly implies that $\mathcal{U}$ is an open covering of K. Since K is compact and nonempty there is a finite subfamily $\mathcal{F}$ of $\mathcal{U}$ such that $K \subset \bigcup\mathcal{F}$. By the definition of $\mathcal{U}$ there is for each $U \in \mathcal{F}$ an open set $v(U)$ such that $y \in v(U)$ and $U \cap v(U) = \emptyset$. Thus $\bigcap\{v(U) : U \in \mathcal{F}\}$ is an open set containing y and disjoint from K. That is, y is an interior point of $X \sim K$. We conclude that $X \sim K$ is open so K is closed. □

Theorem 1.15. Suppose K is compact, $F \subset K$ and F is closed. Then F is compact.

Proof. Suppose $\mathcal{U}$ is an open covering of F. Then $\mathcal{V} = \{X \sim F\} \cup \mathcal{U}$ is an open covering of K. As K is compact, there is a finite subfamily $\mathcal{F}$ of $\mathcal{V}$ such that $K \subset \bigcup\mathcal{V}$. Let $\mathcal{F} = \mathcal{V} \sim \{X \sim K\}$. Evidently, $\mathcal{F}$ is a finite subfamily of $\mathcal{U}$ and $F \subset \cup\mathcal{F}$. □

Definition 1.13. A family $\mathcal{Z}$ of sets has the **finite intersection property** if $\bigcap\mathcal{F} \neq \emptyset$ whenever $\mathcal{F}$ is a nonempty finite subfamily of $\mathcal{Z}$.

Theorem 1.16. Suppose F is a closed subset of X. Then F is compact if and only if $\cap\mathcal{Z} \neq \emptyset$ whenever $\mathcal{Z}$ is a nonempty family of closed subsets of F with the finite intersection property.

Proof. Suppose F is compact and $\mathcal{Z}$ is a nonempty family of closed subsets of F with the finite intersection property. Were it the case that $\bigcap\mathcal{Z} = \emptyset$ we could set $\mathcal{U} = \{X \sim Z : Z \in \mathcal{Z}\}$ obtaining an open covering of F. Since F is compact there would be a finite family $\mathcal{F}$ of $\mathcal{Z}$ such that $F \subset \bigcup\{X \sim Z : Z \in \mathcal{F}\}$. But then

$$\bigcap\mathcal{F} = X \sim \bigcup\{X \sim Z : Z \in \mathcal{F}\} \subset X \sim F,$$

which implies $\bigcap \mathcal{F} = \emptyset$, contradicting that $\mathcal{Z}$ has the finite intersection property.

On the other hand, suppose that $\cap\mathcal{Z} \neq \emptyset$ whenever $\mathcal{Z}$ is a nonempty family of closed subsets of F with the finite intersection property. Let $\mathcal{U}$ be a nonempty open covering of F. Then $\mathcal{Z} = \{F \sim U : U \in \mathcal{U}\}$ is a family of closed subsets of F and $\cap\mathcal{Z} = \emptyset$. Thus there is a finite subfamily $\mathcal{F}$ of $\mathcal{U}$ such that $\bigcap\{F \sim Z : Z \in \mathcal{F}\} = \emptyset$ which implies $F \subset \bigcup \mathcal{F}$. Thus F is compact. □

Theorem 1.17. Suppose $a, b \in \mathbb{R}$ and $a < b$. Then $[a, b]$ is compact.

Proof. Let $\mathcal{U}$ be an open covering of $[a, b]$. Let

$$T = \{t \in (a, b] : \text{ there is a finite subfamily } \mathcal{F} \text{ of } \mathcal{U} \text{ such that } [a, t) \subset \cup\mathcal{F}\}.$$

Since $a \in [a, b]$ there is $U \in \mathcal{U}$ such that $a \in U$. Since U is open there is $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subset U$. Thus $a < \min\{a + \epsilon, b\} \in T$. Also, b is an upper bound for T. So if we let $u = \sup T$ we find that $a < u \leq b$.

Since $u \in [a, b]$ there is $V \in \mathcal{U}$ such that $u \in V$. Since V is open there is $\eta > 0$ such that $(u - \eta, u + \eta) \subset V$. Since $a < u$ there is $t \in (u - \eta, u] \cap T$. Thus there is a finite subfamily $\mathcal{G}$ of $\mathcal{U}$ such that $[a, t) \subset \cup\mathcal{G}$. Let $\mathcal{F} = \mathcal{G} \cup \{V\}$. Then $\mathcal{F}$ is a finite subfamily of $\mathcal{U}$ and $[a, u + \eta) \subset \cup\mathcal{F}$. Were it the case that $u < b$ we would have $u < v = \min\{u + \eta, b\} \leq b$ which would imply $v \in T$ which is incompatible with u being an upper bound for T. Thus $u = b$ and $[a, b] \subset \mathcal{F}$, as desired. □

1.5. Continuity.

Definition 1.14. Suppose X and Y are topological spaces, $A \subset X$ and

$$f : A \to Y.$$

We say f is **continuous** if for each open subset V of Y such that there is an open subset U of X such that

$$f^{-1}[V] = U \cap A.$$

We leave as an exercise for the reader the straightforward verification of the fact that f is continuous if and only if for each closed subset F of Y there is a closed subset E of X such that

$$f^{-1}[F] = A \cap E.$$

Theorem 1.18. Suppose X and Y are topological spaces, $A \subset X$ and

$$f : A \to Y.$$

Then f is continuous if and only if f is continuous with respect to the relative topology for A.

Proof. This is an immediate consequence of the definitions. □

Theorem 1.19. Suppose X, Y and Z are topological spaces, $A \subset X$, $B \subset Y$ and

$$f : A \to Y, \quad \text{and} \quad g : B \to Z$$

are continuous. Then

$$g \circ f : A \cap f^{-1}[B] \to Z$$

is continuous.

Proof. Straightforward exercise for the reader. □

Theorem 1.20. Suppose X and Y are topological spaces, $A \subset X$ and

$$f : A \to Y.$$

is continuous. Then

(i) if A is connected then $f[A]$ is connected;
(ii) if A is compact then $f[A]$ is compact.

Proof. We prove (i) and leave the proof of (ii) as an exercise for the reader. We give two proofs of (i). **Proof One.** Suppose E and F are open subsets of Y such that $f[A] \subset E \cup F$ and $f[A] \cap E \cap F = \emptyset$. We need to show that *either* (i) $f[A] \subset E$ *or* (ii) $f[A] \subset F$.

Since f is continuous there are open subsets U and V of X such that $f^{-1}[U] = A \cap E$ and $f^{-1}[V] = A \cap F$. Now

$$A \subset f^{-1}[f[A]] \subset f^{-1}[E \cup F] = f^{-1}[E] \cup f^{-1}[F] = (A \cap U) \cup (A \cup V) \subset U \cup V$$

and, keeping in mind that $A \subset f^{-1}(Z)$ for any set Z,

$$A \cap U \cap V = (A \cap U) \cap (A \cap V) = f^{-1}[E] \cap f^{-1}[F] = f^{-1}[E \cap F] = f^{-1}[f[A] \cap E \cap F] = \emptyset.$$

Since A is connected we have *either* $A \subset U$ in which case (i) holds *or* $A \subset V$ in which case (ii) holds. **Proof Two.** Let $\mathcal{T}$ be the topology for X and let $\mathcal{U}$ be the topology for Y. We leave as a straightforward exercise for the reader the proof of the statement that that

$$f \text{ is } (A, \mathcal{T}_A) - (f[A], \mathcal{U}_{f[A]}) \text{ continuous.}$$

Suppose that G and H are $\mathcal{U}_{f[A]}$-open, $f[A] = G \cup H$ and $G \cap H = \emptyset$. Then $f^{-1}[G]$ and $f^{-1}[H]$ are $\mathcal{T}_A$-open, $f^{-1}[G] \cup f^{-1}[H] = f^{-1}[G \cup H] \subset f^{-1}[f[A]] = A$ and $f^{-1}[G] \cap f^{-1}[H] = f^{-1}[G \cap H] = f^{-1}[\emptyset] = \emptyset$. Since A is $\mathcal{T}_A$ connected we have *either* (i) $A \subset f^{-1}[G]$ in which case $f[A] \subset f[f^{-1}[G]] = G$ *or* $A \subset f^{-1}[H]$ in which case $f[A] \subset f[f^{-1}[H]] = H$. Thus $f[A]$ is connected.

□

1.6. Limits. We suppose throughout this section that X and Y are topological spaces, $A \subset X$ and

$$f : A \to Y.$$

Definition 1.15. Suppose $a \in \textbf{acc}\, A$ and $b \in Y$. We say $f(x)$ **has b as a limit as x approaches a** and write

$$\lim_{x \to a} f(x) = b$$

if for each open subset V of Y such that $b \in V$ there is an open subset U of X such that $a \in U$ and $A \cap (U \sim \{a\}) \subset f^{-1}[V]$.

If $a \in A$ we say f **is continuous at** a if either $a \in \textbf{iso}\, A$ or $a \in \textbf{acc}\, A$ and $\lim_{x \to a} f(x) = f(a)$.

Exercise 1.2. Suppose $a \in \textbf{acc}\, A$. Show that

$$\{b \in Y : \lim_{x \to a} f(x) = b\}$$

is a closed subset of Y.

Theorem 1.21. Suppose Y is Hausdorff, $a \in \mathbf{acc}\, A$ and

$$\lim_{x \to a} f(x) = b_i \in Y, \quad i = 1, 2.$$

Then

$$b_1 = b_2.$$

Proof. Suppose, contrary to the Theorem, $b_1 \neq b_2$. For each $i = 1, 2$ let V_i be an open subset of Y such that $b_i \in V_i$ and $V_1 \cap V_2 = \emptyset$. Then for each $i = 1, 2$ there is an open subset U_i of X such that $a \in U_i$ and

$$A \cap (U_i \sim \{a\}) \subset f^{-1}[V_i].$$

Since $a \in \mathbf{acc}\, A$ we have

$$\begin{aligned} \emptyset &\neq A \cap ((U_1 \cap U_2) \sim \{a\}) \\ &= (A \cap (U_1 \sim \{a\})) \cap (A \cap (U_2 \sim \{a\})) \\ &\subset f^{-1}[V_1] \cap f^{-1}[V_2] \\ &= f^{-1}[V_1 \cap V_2] \\ &= \emptyset. \end{aligned}$$

□

Theorem 1.22. Suppose $A \subset \mathbb{R}n$, $a \in \mathbf{acc}\, A$,

$$f : A \to \mathbb{R}m$$

and $b \in \mathbb{R}m$.

The following are equivalent.

(i) $\lim_{x \to a} f(x) = b$.

(ii) For each $\epsilon > 0$ there is $\delta > 0$ such that

$$x \in A \sim \{a\} \text{ and } |x - a| < \delta \;\Rightarrow\; |f(x) - b| < \epsilon.$$

Proof. Suppose (i) holds and $\epsilon > 0$. Let $V = \mathbf{U}^b(\epsilon)$. Since V is an open subset of $\mathbb{R}$ there is an open subset U of $\mathbb{R}$ such that $a \in U$ and $A \cap (U \sim \{a\}) \subset f^{-1}[V]$; thus

$$x \in A \cap (U \sim \{a\}) \;\Rightarrow\; f(x) \in V.$$

Since $a \in U$ and U is open there is $\delta > 0$ such that $\mathbf{U}^a(\delta) \subset U$. If $x \in A \sim \{a\}$ and $|x - a| < \delta$ then $x \in A \cap (U \sim \{a\})$ so $f(x) \in V$ so $|f(x) - b| < \epsilon$.

Suppose (ii) holds, V is an open subset of $\mathbb{R}$ and $b \in V$. Since V is open there is $\epsilon > 0$ such that $\mathbf{U}^b(\epsilon) \subset V$. Let $\delta > 0$ be such that

$$x \in A \sim \{a\} \text{ and } |x - a| < \delta \;\Rightarrow\; |f(x) - b| < \epsilon.$$

Let $U = \mathbf{U}^a(\delta)$. If $x \in A \cap (U \sim \{a\})$ then $x \in A \sim \{a\}$ and $|x - a| < \delta$ so $|f(x) - b| < \epsilon$ so $f(x) \in V$. Thus $A \cap (U \sim \{a\}) \subset f^{-1}[V]$ so (i) holds. □

Theorem 1.23. f is continuous if and only if f is continuous at a for each $a \in A \cap \mathbf{acc}\, A$.

Proof. Straightforward exercise for the reader. □

The following Theorem is extremely useful.

Theorem 1.24. Suppose $a \in \mathbf{acc}\, A$, $b \in Y$, $B \subset Y$, $b \in \mathbf{int}\, B$, Z is a topological space, $g : B \to Z$,

$$\lim_{x \to a} f(x) = b \quad \text{and } g \text{ is continuous at } b.$$

Then $\mathbf{dmn}\,(g \circ f) = A \cap f^{-1}[B]$, a is an accumulation point for $\mathbf{dmn}\,(g \circ f)$ and

$$\lim_{x \to a} g \circ f(x) = g(b).$$

Proof. Exercise for the reader. $\square$