Homework One.

1. Part One.

1. From **Sets, relations and functions** do Exercises 1.2, 1.3 and 1.5.

2. Prove Theorem 1.1 in **Sets, relations and functions**.

2. Part Two. Limits.

Do the two exercises which appear below.

Definition 2.1. Suppose $A \subset \mathbb{R}$ and $a \in \mathbb{R}$. We say a **is an accumulation point of** A if

$$A \cap \{x \in \mathbb{R} : 0 < |x-a| < \delta\} \neq \emptyset \quad \text{whenever } 0 < \delta < \infty.$$

Definition 2.2. Suppose $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$, a is an accumulation point of a and $L \in \mathbb{R}$. Then

$$\lim_{x \to a} f(x) = L$$

if for each $\epsilon > 0$ there is $\delta > 0$ such that

$$x \in A \text{ and } 0 < |x-a| < \delta \;\Rightarrow\; |f(x) - L| < \epsilon.$$

Example 2.1. Suppose $f(x) = x^2$ for $x \in \mathbb{R}$. Then for any $a \in \mathbb{R}$ a is an accumulation point of $\mathbb{R}$ (obviously, right?) and we have

$$\lim_{x \to a} f(x) = f(a).$$

Here is a proof of this statement. Suppose $a \in \mathbb{R}$ and $\epsilon > 0$. Suppose $x \in \mathbb{R}$ and $0 < |x-a| < \delta \leq 1$ we have

$$|f(x) - f(a)| = |x^2 - a^2| = |(x-a) + 2a||x-a| \leq (|x-a| + 2|a|)|x-a| \leq (1 + 2|a|)\delta.$$

This last quantity will be less than ϵ if

$$\delta < \frac{\epsilon}{1 + 2|a|}.$$

Exercise 2.1. Now let $f = \{(x, 1/x) : x \in \mathbb{R} \sim \{0\}\}$. Thus $f : \mathbb{R} \sim \{0\} \to \mathbb{R}$.

Suppose $a \in \mathbb{R} \sim \{0\}$. I want you to prove that a is an accumulation point of $\mathbb{R} \sim \{0\}$ and that

$$\lim_{x \to a} f(x) = f(a).$$

Note that if $x \in \mathbb{R} \sim \{0\}$ then

$$|f(x) - f(a)| = \left| \frac{1}{x} - \frac{1}{a} \right| = \frac{|x-a|}{|x||a|}.$$

Example 2.2. Let $f = ((-\infty, 0) \times \{0\}) \cup ([0, \infty) \times \{1\})$. Then $f : \mathbb{R} \to \mathbb{R}$ and

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } 0 \leq x \end{cases}$$

for any $x \in \mathbb{R}$.

We have already noted that 0 is an accumulation point of $\mathbb{R}$. We will prove that

$$\lim_{x \to 0} f(x) = L$$

for no $L \in \mathbb{R}$.

We need to prove that for any $L \in \mathbb{R}$ there is $\epsilon > 0$ such that for any $\delta > 0$ there is x with $0 < |x| < \delta$ such that $|f(x) - L| \geq \epsilon$.

Suppose $L \in \mathbb{R}$, $0 < \epsilon \leq 1/2$ and $0 < \delta < \infty$. If $L \geq 1/2$ we have

$$a \in (-\delta, 0) \ \Rightarrow |f(a) - L| = |L| \geq 1/2$$

and if $L < 1/2$ we have

$$b \in (0, \delta) \ \Rightarrow |f(b) - L| = |1 - L| \geq 1/2.$$

Exercise 2.2. Let f be as in Exercise 2.1. Prove that

$$\lim_{x \to 0} f(x) = L$$

for no $L \in \mathbb{R}$.