1. The integers and the rational numbers.

1.1. The integers. Let

$$\mathcal{Z} = \{((m,n),(p,q)) \in (\mathbb{N}\times\mathbb{N})\times(\mathbb{N}\times\mathbb{N}) : m+q = p+n\}.$$

It is a simple matter to verify that $\mathcal{Z}$ is an equivalence relation on $\mathbb{N}\times\mathbb{N}$. Let

$$\mathbb{Z} = (\mathbb{N}\times\mathbb{N})/\mathcal{Z}$$

and call its members **integers**. Let

$$m-n = (m,n)/\mathcal{Z} \quad \text{for } (m,n) \in \mathcal{N}\times\mathcal{N}.$$

One easily verifies that

$$\mathbb{N} \ni n \mapsto (n-0)/\mathcal{Z}$$

is univalent. In what follows we will not distinguish between a member of $\mathbb{N}$ and its image under this mapping.

One verifies that there is a unique unary operation $-$ on $\mathbb{Z}$ such that

$$-(m-n) = n-m$$

and that there are unique binary operation $+$ and $*$ on $\mathbb{Z}$ such that

$$(m-n)+(p-q) = (m+p)-(n+q) \quad \text{and} \quad (m-n)*(p-q) = (m*p+n*q)-(m*q+n*p)$$

for $(m,n),(p,q) \in \mathbb{N}\times\mathbb{N}$; on the right hand side of these equations $+$ and $*$ are the operations on $\mathbb{N}$.

One easily verifies that $(\mathbb{Z},+,0,-)$ is an Abelian group and that $(\mathbb{Z},+,0,*)$ is an integral domain in which 1 is the neutral element with respect to $*$.

1.2. The rational numbers. Let

$$\mathcal{Q} = \{((m,n),(p,q)) \in (\mathbb{Z}\times(\mathbb{Z}\sim\{0\}))\times(\mathbb{Z}\times(\mathbb{Z}\sim\{0\})) : mq = np\}.$$

It is a simple matter to verify that $\mathcal{Q}$ is an equivalence relation on $\mathbb{Z}\times(\mathbb{Z}\sim\{0\})$. Let

$$\mathbb{Q} = (\mathbb{Z}\times(\mathbb{Z}\sim\{0\}))/\mathcal{Q}$$

and call its members **rational numbers**. Let

$$\frac{m}{n} = (m,n)/\mathcal{Q} \quad \text{for } (m,n) \in \mathcal{Z}\times(\mathcal{Z}\sim\{0\}).$$

One easily verifies that

$$\mathbb{Z} \ni n \mapsto \left(\frac{n}{1}\right)/\mathcal{Q}$$

is univalent. In what follows we will not distinguish between a member of $\mathbb{Z}$ and its image under this mapping.

One verifies that there is a unique unary operation $-$ on $\mathbb{Z}$ such that

$$-\frac{m}{n} = \frac{-m}{n} \quad \text{for } (m,n) \in \mathbb{Z}\times(\mathbb{Z}\sim\{0\})$$

and that there are unique binary operation $+$ and $*$ on $\mathbb{Z}$ such that

$$\frac{m}{n}+\frac{p}{q} = \frac{mq+np}{nq} \quad \text{and} \quad \frac{m}{n}*\frac{p}{q} = \frac{m*p}{n*q}$$

for $(m,n),(p,q) \in \mathbb{N}\times\mathbb{N}$; on the right hand side of these equations $-$, $+$ and $*$ are the operations on $\mathbb{Z}$.

One easily verifies that and that $(\mathbb{Q},+,0,*,1)$ is an field.

Note that if one replaces $\mathbb{Z}$ by any integral domain D this construction results in a field which is called the **field of quotients of** D.