1. Power series.

Definition 1.1. Suppose c is a sequence in $\mathbf{C}$. (c will be a *coefficient* sequence.) We set

$$M(c,r) = \sup\{|c_n|r^n : n \in \mathbf{N}\} \quad \text{whenever } 0 \le r < \infty$$

and we let

$$R(c) = \sup\{r \in [0,\infty) : M(c,r) < \infty\}.$$

Note that $M(c,r)$ and $R(c)$ could equal ∞. We call $R(c)$ the **radius of convergence of** c for reasons which will shortly become apparent.

We repeatedly use the following estimate:

$$\sum_{n=N}^{\infty} |c_n||z-z_0|^n \le M(c,r)\Big(\frac{|z-z_0|}{r}\Big)^N \frac{r}{r-|z-z_0|} \qquad \text{whenever } |z-z_0| < r < R(c) \text{ and } N \in \mathbf{N}. \tag{1}$$

Indeed, if S is the left hand side of (1) we have

$$\begin{aligned} S &= \sum_{n=N}^{\infty} |c_n| r^n \Big(\frac{|z-z_0|}{r}\Big)^n \\ &\le M(c,r) \sum_{n=N}^{\infty} \Big(\frac{|z-z_0|}{r}\Big)^n \\ &= M(c,r)\Big(\frac{|z-z_0|}{r}\Big)^N \frac{r}{r-|z-z_0|}. \end{aligned}$$

Proposition 1.1. Suppose c is a sequence in $\mathbf{C}$ and $z_0 \in \mathbf{C}$. Then for any $z \in \mathbf{C}$ we have

$$|z-z_0| < R(c) \;\Rightarrow\; \sum_{n=0}^{\infty} |c_n(z-z_0)^n| < \infty;$$

$$\sum_{n=0}^{\infty} |c_n(z-z_0)^n| < \infty \;\Rightarrow\; |z-z_0| \le R(c);$$

and

$$|z-z_0| > R(c) \;\Rightarrow\; \limsup_{n\to\infty} |c_n(z-z_0)^n| = \infty \;\Rightarrow\; \sum_{n=0}^{\infty} |c_n(z-z_0)^n| = \infty.$$

Proof. The first and second inferences follow directly from (1) and the third follows directly from the definition of $M(c,|z-z_0|)$. $\square$

Definition 1.2. Suppose $A \subset \mathbf{C}$ and

$$f : A \to \mathbf{C}.$$

We let

$$f' = \{(a,m) : a \in \textbf{int}\, A \text{ and } m = \lim_{z\to a} \frac{f(z)-f(a)}{z-a}\}.$$

Note that f' is a function. We say f is **differentiable at** a if a is in the domain of f'. For each nonnegative integer m we define $f^{(m)}$ by setting $f^{(0)} = f$, $f^{(1)} = f'$ and requiring that $f^{(m+1)} = (f^{(m)})'$. This notion of differentiability is *very* different from the previous notion of differentiability although that may not be obvious.

Lemma 1.1. Suppose c is a sequence in $\mathbf{C}$, $z_0 \in \mathbf{C}$ and

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad \text{for } |z - z_0| < R(c). \tag{1}$$

Then

$$f'(z_0) = c_1.$$

Proof. Suppose $0 < |z - z_0| < r < R(c)$. We have

$$\begin{aligned} |f(z) - f(z_0) - c_1(z - z_0)| &= |\sum_{n=2}^{\infty} c_n (z - z_0)^n| \\ &\leq \sum_{n=2}^{\infty} |c_n||z - z_0|^n \\ &\leq M(c, r)\Big(\frac{|z - z_0|}{r}\Big)^2 \frac{r}{r - |z - z_0|} \end{aligned}$$

so

$$\Big|\frac{f(z) - f(z_0)}{z - z_0} - c_1| \leq M(c, r)\frac{|z - z_0|}{r^2}\frac{r}{r - |z - z_0|}.$$

□

Proposition 1.2. Suppose c and d are sequences in $\mathbf{C}$, $M \in \mathbf{Z}$, $N \in \mathbf{N}$, $M + N \geq 0$ and

$$c_n = d_{M+n}, \text{ whenever } n \geq N.$$

Then

$$R(c) = R(d).$$

Proof. Suppose $0 \leq r < R(d)$. If $n \in \mathbf{N}$ and $n \geq N$ we have

$$|c_n|r^n = r^{-M}|d_{M+n}|r^{M+n} \leq r^{-M}M(d, r) < \infty$$

which implies that $M(c, r) < \infty$ so $R(c) \geq R(d)$.

Suppose $0 \leq R(c)$. If $n \in \mathbf{N}$ and $n \geq N$ then

$$|d_{M+n}|r^{n+M} = r^M|c_n|r^n \leq r^M M(c, r) < \infty$$

which implies that $M(d, r) < \infty$ so $R(d) \leq R(c)$. □

Exercise 1.1. Prove the following statements.

(i) Suppose c is a sequence in $\mathbf{C}$ and $a \in \mathbf{C} \sim \{0\}$. Then $R(ac) = R(c)$.

(ii) Suppose b and c are sequences in $\mathbf{C}$. Then $\min\{R(b), R(c)\} \leq R(b + c)$.

Exercise 1.2. (The ratio test.) Suppose c is a sequence in $\mathbf{C} \sim \{0\}$ and

$$L = \limsup_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} < \infty.$$

Show that

$$\frac{1}{L} \leq R(c).$$

Proposition 1.3. (The root test.) Suppose c is a sequence in $\mathbf{C}$ and

$$L = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}.$$

Then

$$R(c) = \begin{cases} \infty & \text{if } L = 0, \\ \frac{1}{L} & \text{if } 0 < L < \infty, \\ 0 & \text{else.} \end{cases}$$

Proof. The point here is that

$$|c_n| r^n = \left(|c_n|^{\frac{1}{n}} r\right)^n,\ n \in \mathbf{N},\ 0 \le r < \infty.$$

We leave the straightforward details which remain to the reader. □

Exercise 1.3. Suppose c is a sequence in $\mathbf{C}$ and N is a nonnegative integer such that

$$c_n = 0 \quad \text{if} \quad n < N \quad \text{and} \quad c_N \neq 0.$$

show that for each $\lambda \in (0,1)$ there is $\delta \in (0, R(c))$ such that

$$\Big|\sum_{n=0}^{\infty} c_n (z - z_0)^n\Big| \ge \lambda |c_N| |z - z_0|^N \text{provided } |z - z_0| < \delta.$$

Hint: This follows easily from (1).

Exercise 1.4. Suppose c is a sequence in $\mathbf{C}$ and $z_0, z_1, z \in \mathbf{C}$ are such that

$$|z_1 - z_0| + |z - z_1| < R(c).$$

Make sense of the following:

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n (z - z_0)^n &= \sum_{n=0}^{\infty} c_n [(z_1 - z_0) + (z - z_1)]^n \\
&= \sum_{n=0}^{\infty} c_n \Big(\sum_{m=0}^{n} \binom{n}{m} (z_1 - z_0)^{n-m} (z - z_1)^m \Big) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} c_n \binom{n}{m} (z_1 - z_0)^{n-m} (z - z_1)^m \\
&= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} c_n \binom{n}{m} (z_1 - z_0)^{n-m} (z - z_1)^m \\
&= \sum_{n=0}^{\infty} \Big(\sum_{m=n}^{\infty} c_m \binom{m}{n} (z_1 - z_0)^{m-n} \Big) (z - z_1)^n.
\end{aligned}$$

Hint: Use the ideas in the proof that $e^{z+w} = e^z e^w$ for $z, w \in \mathcal{C}$.

Theorem 1.1. Suppose c is a sequence in $\mathbf{C}$, $z_0 \in \mathbf{C}$, $R(c) > 0$, $D = \{z \in \mathbf{C} : |z - z_0| < R(c)\}$ and

$$f : D \to \mathbf{C}$$

is such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad \text{for } z \in D.$$

Then for any positive integer m the domain of $f^{(m)}$ equals D and

$$f^{(m)}(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} c_n (z-z_0)^{n-m} \quad \text{whenever } z \in D.$$

Proof. From the preceding we have

$$f(z) = \sum_{n=0}^{\infty} \Big(\sum_{m=n}^{\infty} c_m \binom{m}{n} (z_1 - z_0)^{m-n} \Big) (z - z_1)^n \quad \text{whenever } z_1 \in D.$$

From a preceding Proposition we infer that

$$f'(z_1) = \sum_{m=1}^{\infty} c_m \binom{m}{1} (z_1 - z_0)^{m-1} \quad \text{whenever } z_1 \in D.$$

Replacing z_1 by z and m by n we find that

$$f'(z) = \sum_{n=1}^{\infty} c_n \binom{n}{1} (z - z_0)^{n-1} \quad \text{whenever } z \in D.$$

Substituting $n+1$ for n we obtain

$$f'(z) = \sum_{n=0}^{\infty} c_{n+1} (n+1) (z - z_0)^{n} \quad \text{whenever } z \in D$$

thereby establishing the Theorem when $m = 1$. Now induct. □

Remark 1.1. Note that *no estimation was required!*.

Remark 1.2. Note that a consequence of the foregoing is that for any $m \in \mathbf{N}$ the radius of convergence of

$$\mathbf{N} \ni n \mapsto \frac{(n+m)!}{n!} c_{n+m} \in \mathbf{C}$$

equals $R(c)$. One could also prove this directly from the definition by a slightly tricky argument or one could deduce it from the root test. For yet another proof, one could invoke the fact that if a sequence of functions converges uniformly to a limit F and the sequence of derivatives converges uniformly to a limit G then $F' = G$.

Remark 1.3. Note that all of the foregoing goes through when $\mathbb{C}$ is replaced by $\mathbb{R}$. There are further generalizations as well.

Definition 1.3. Suppose A is an open subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$. We say f is **real analytic** if for each $a \in A$ there is a sequence c in $\mathbb{R}$ such that $\mathbf{R}(c) > 0$ and

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{whenever } x \in A \text{ and } |x - a| < \mathbf{R}(c).$$

Exercise 1.5. Suppose A is an open subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$.

Show that f is real analytic if and only if the domain of $f^{(n)}$ equals A for all $n \in \mathbb{N}$ and for each $a \in I$ there are $\delta > 0$ and $M \in [0, \infty)$ such that

$$\left| f^{(n)}(x) \right| \leq M^n n! \quad \text{whenever } n \in \mathbb{N},\ x \in A \text{ and } |x - a| < \delta.$$

You will need to make use of Taylor's Theorem.

Exercise 1.6. Show that the composition of real analytic functions is real analytic. One way to do this is to come up with a clever inductive scheme to estimate higher derivatives of the composition of two functions.