1. Metric spaces.

Definition 1.1. Let X be a set. We say ρ is a **metric on** X if

$$\rho : X \times X \to \{r \in \mathbb{R} : r \geq 0\}$$

and

(i) $\rho(x, y) = \rho(y, x)$ whenever $x, y \in X$;
(ii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ whenever $x, y, z \in X$.
(iii) If $x, y \in X$ then $\rho(x, y) = 0$ if an only if $x = y$.

The inequality in (ii) is called the **triangle inequality**. A **metric space** is an ordered pair (X, ρ) such that X is a set and ρ is a metric on X.

We now fix a set X and a metric ρ on X.

For each $a \in X$ and each positive real number r we let

$$\mathbf{U}^a(r) = \{x \in X : \rho(x, a) < r\} \text{ and we let } \mathbf{B}^a(r) = \{x \in X : \rho(x, a) \leq r\}.$$

We say a subset U of X is **open** if for each $a \in U$ there is a positive real number ϵ such that $\mathbf{U}^a(\epsilon) \subset U$. We leave as an exercise to the reader the proof of the fact that *the family of open sets is a topology on X*. This topology is called the **topology induced by the metric** ρ; one proves this in exactly the same way we proved the corresponding fact for $\mathbb{R}^n$.

Suppose x is a sequence in X and $b \in X$. Note that

$$\lim_{\nu \to \infty} x_\nu = b$$

if and only if for each $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\rho(x_\nu, b) < \epsilon \quad \text{whenever } \nu \in \mathbb{N} \text{ and } n \geq N.$$

Theorem 1.1. Suppose $a \in X$ and r is a positive real number. Then

$$\mathbf{U}^a(r) \text{ is open and } \mathbf{B}^a(r) \text{ is closed.}$$

Proof. Suppose $b \in \mathbf{U}^a(r)$. I claim that $\mathbf{U}^b(r - \rho(a, b)) \subset \mathbf{U}^a(r)$. Indeed, suppose $x \in \mathbf{U}^b(r - \rho(a, b))$; then, by the triangle inequality and the fact that $\rho(a, b) = \rho(b, a)$,

$$\rho(x, a) \leq \rho(x, b) + \rho(b, a) = \rho(x, a) + \rho(a, b) < (r - \rho(a, b)) + \rho(a, b) = r$$

so $x \in \mathbf{U}^a(r)$. Thus $\mathbf{U}^a(r)$ is open. The reader should verify that, in a similar fashion, one may prove that $X \sim \mathbf{B}^a(r)$ is open so that $\mathbf{B}^a(r)$ is closed. □

Theorem 1.2. Then the topology induced by the metric ρ is Hausdorff.

Proof. Suppose $x, y \in X$ and $x \neq y$. Let $r = \rho(x, y)/2$, note that $r > 0$ and let $U = \mathbf{U}^x(r)$ and $V = \mathbf{U}^y(r)$. Then, by the previous theorem, U and V are open. Suppose $z \in U$. Then, by the triangle inequality and the fact that $\rho(x, z) = \rho(z, x)$, we infer that

$$\rho(z, y) \geq \rho(x, y) - \rho(x, z) = \rho(x, y) - \rho(z, x) > r - r/2 = r/2$$

so $z \notin V$. Thus $U \cap V = \emptyset$ and this proves that X is Hausdorff. □

Definition 1.2. Whenever $A \subset X$ and $x \in X$ we let

$$\rho(a, A) = \inf\{\rho(x, y) : y \in A\}$$

and we call this number the **distance from** a **to** A.

Theorem 1.3. Suppose A is a subset of X. Then
(i) $|\rho(x, A) - \rho(y, A)| \le \rho(x, y)$ whenever $x, y \in X$;
(ii) $\mathbf{cl}\, A = \{x \in X : \rho(x, A) = 0\}$;
(iii) $\mathbf{int}\, A = \{x \in X : \rho(x, X \sim A) > 0\}$.

Proof. We leave this as an exercise to the reader. In proving (i) one makes use of the fact that if a, b and c are real numbers then

$$|a - b| \le c \;\Leftrightarrow\; a \le b + c \text{ and } b \le a + c$$

which implies that

$$||a| - |b|| \le |a - b|.$$

□

Definition 1.3. Suppose A is a subset of X. We let

$$\mathbf{diam}\, A = \sup\{\rho(x, y) : x, y \in A\}$$

and call this number the **diameter of** A. We say A is **bounded** if $\mathbf{diam}\, A < \infty$.

2. Completeness.

Definition 2.1. We say (X, ρ) is **complete** (or when it is clear from the context what ρ is that X **is complete**) if

$$\bigcap \mathcal{C} \neq \emptyset$$

whenever $\mathcal{C}$ is a nonempty nested family of nonempty closed subsets of X such that

$$\inf\{\mathbf{diam}\, C : C \in \mathcal{C}\} = 0.$$

Note that if $\mathcal{C}$ is as in the preceding definition then $\bigcap \mathcal{C}$ has exactly one point.

A sequence x in X is a **Cauchy sequence** if

$$\inf\{\mathbf{diam}\, \{x_m : m \in \mathbb{N} \text{ and } m \ge n\} : n \in \mathbb{N}\} = 0.$$

This is equivalent to the statement that for each positive real number ϵ there is a nonnegative integer N such that

$$\rho(x_l, x_m) \le \epsilon \text{ whenever } l \ge N \text{ and } m \ge N.$$

Proposition 2.1. X is complete if and only if every Cauchy sequence converges.

Proof. Suppose X is complete and x is a Cauchy sequence in X. For each positive integer m let $C_m = \mathbf{cl}\, \{x_n : m \in \mathbb{N} \text{ and } m \ge n\}$ and note that $\mathcal{C} = \{C_m : m \in \mathbb{N}$ is a nonempty nested family of closed subsets of X with the property that

$$\inf\{\mathbf{diam}\, C : C \in \mathcal{C}\} = 0.$$

Because X is complete there is a unique member l of $\bigcap \mathcal{C}$. We now show that l is the limit of the sequence x. Let $\epsilon > 0$. Chooose $N \in \mathbb{N}$ such that $\mathbf{diam}\, C_N \le \epsilon$. If $n \ge N$ then both l and x_n are members of C_n which is a subset of C_N so $\rho(x_n, l) \le \mathbf{diam}\, C_n \le \mathbf{diam}\, C_N \le \epsilon$.

On the other hand, suppose X is a metric space in which every Cauchy sequence converges and let $\mathcal{C}$ be a nonempty nested family of nonempty closed sets with the property that

$$\inf\{\mathbf{diam}\, C : C \in \mathcal{C}\} = 0.$$

In case there is $C \in \mathcal{C}$ such that $\mathbf{diam}\, C = 0$ then there is $c \in X$ such that $C = \{c\}$ so $\cap \mathcal{C} = \{c\}$. So suppose $\mathbf{diam}\, C > 0$ for each $C \in \mathcal{C}$. Choose a sequence C in $\mathcal{C}$ such that $\mathbf{diam}\, C_{\nu+1} < \mathbf{diam}\, C_\nu$ whenever $\nu \in \mathbb{N}$ and such that $\lim_{\nu\to\infty} \mathbf{diam}\, C_\nu = 0$. Note that $C_{\nu+1} \subset C_\nu$ whenever $\nu \in \mathbb{N}$ for if this were not the case for some $\nu \in \mathbb{N}$ we would have $C_\nu \subset C_{\nu+1}$ since $\mathcal{C}$ is nested and this would imply $\mathbf{diam}\, C_\nu \le \mathbf{diam}\, C_{\nu+1}$. Let x be a sequence in X such that $x_\nu \in C_\nu$ for each $m \in \mathbb{N}$. Note that we have used the Axiom of Choice twice. Evidently, x is a Cauchy sequence. Let c be its limit and suppose $B \in \mathcal{C}$. Choose $\nu \in \mathbb{N}$ such that $\mathbf{diam}\, C_\nu < \mathbf{diam}\, B$. For any $\mu \in \mathbb{N}$ with $\mu \ge \nu$ we have $C_\mu \subset C_\nu \subset B$ so, by a preceding Theorem

$$\mathbf{dist}\,(c, B) \le \mathbf{dist}\,(c, C_\mu) \le \rho(c, x_\mu) + \mathbf{dist}\,(x_\mu, C_\mu) = \rho(c, x_\mu) \to 0 \quad \text{as} \quad \nu \to \infty.$$

Thus $\mathbf{dist}\,(c, B) = 0$ so, again by a preceding Theorem, $c \in \mathbf{cl}\, B = B$ Thus $c \in \cap\mathcal{C}$. □

Proposition 2.2. Suppose $A \subset X$ and $\sigma = \rho|(A \times A)$. Then (A, σ) is complete if and only if A is a closed subset of X.

Proof. Suppose (A, σ) is complete. Let b be a point of the ρ-closure of A. For each $\epsilon > 0$ let $C_\epsilon = A \cap \{x \in X : \rho(x, b) \le \epsilon\}$ and note that C_ϵ is σ-closed. Moreover,

$$\emptyset \ne A \cap \{x \in X : \rho(x, b) < \epsilon\} \subset A \cap C_\epsilon \quad \text{and} \quad \mathbf{diam}\, C_\epsilon \le 2\epsilon \quad \text{whenever } 0 < \epsilon < \infty.$$

Thus $\mathcal{C} = \{C_\epsilon : 0 < \epsilon < \infty\}$ is a nonempty family of nonempty σ-closed sets; thus there is $c \in A$ such that $\{c\} = \cap\mathcal{C}$. It is evident that $b = c$ so $b \in A$ and, therefore, A is ρ-closed.

Suppose A is ρ-closed. Let x be a Cauchy sequence in A. Evidently, x is a Cauchy sequence in X. As (X, ρ) is complete there is $b \in X$ such that $\lim_{\nu\to\infty} x_\nu = b$. Since A is ρ-closed we infer that $b \in A$. Thus (A, σ) is complete. □

Theorem 2.1. $\mathbb{R}^n$ is complete.

Proof. We have already proved this in the case $n = 1$.

Suppose x is a Cauchy sequence in $\mathbb{R}^n$. For each $i \in \{1, \ldots, n\}$ let $p_i : \mathbb{R}^n \to \mathbb{R}$ assign to $a \in \mathbb{R}^n$ its ith coordinate; note that $|p_i(a)| \le |a|$ whenever $a \in \mathbb{R}^n$. This implies $p_i \circ x$ is a Cauchy sequence in $\mathbb{R}$ for each $i \in \{1, \ldots, n\}$ which, therefore, converges to some $L_i \in \mathbb{R}$. Let $L \in \mathbb{R}^n$ be such that $p_i(L) = L_i$ for $i \in \{1, \ldots, n\}$. Then

$$|x_\nu - L| \le \sqrt{n} \max\{|p_i(x) - p_i(L)| : i \in \{1, \ldots, n\}\} \to 0 \quad \text{as } \nu \to \infty.$$

That is, $\lim_{\nu\to\infty} x_\nu = L$. □

3. The Lebesgue radius of an open covering.

Definition 3.1. Suppose $\mathcal{U}$ is a family of open subsets of X. X. For each $x \in \cup\mathcal{U}$ we let

$$\iota_{\mathcal{U}}(x) = \{r : 0 < r < \infty \text{ and } \mathbf{U}^x(r) \subset U \text{ for some } U \in \mathcal{U}\};$$

evidently $\iota_{\mathcal{U}}(x)$ is a nonempty open interval with infimum 0. For each $x \in \cup\mathcal{U}$ we let

$$\rho_{\mathcal{U}}(x) = \sup \iota_{\mathcal{U}}(x)$$

and note that $0 < \rho_{\mathcal{U}}(x) \le \infty$. We let

$$l_{\mathcal{U}} = \inf\{\rho_{\mathcal{U}}(x) : x \in X\}.$$

We call $l_{\mathcal{U}}$ the **Lebesgue radius of** $\mathcal{U}$. Evidently,

$$0 < r < l_{\mathcal{U}} \;\Leftrightarrow\; \text{for each } a \in X \text{ there is } U \in \mathcal{U} \text{ such that } \mathbf{U}^a(r) \subset U.$$

Lemma 3.1. Suppose $\mathcal{U}$ is an open covering of X and $0 < s < \infty$. Then $\{x \in X : \rho_{\mathcal{U}}(x) > s\}$ is open.

Proof. Let $G = \{x \in X : \rho_{\mathcal{U}}(x) > s\}$ and suppose $a \in G$. Choose t, u such that $s < t < u < \rho_{\mathcal{U}}(a)$. Next, choose $U \in \mathcal{U}$ such that $\mathbf{U}^a(u) \subset U$.

Suppose $x \in \mathbf{U}^a(u - t)$. Then

$$\mathbf{U}^x(t) \subset \mathbf{U}^a(u) \subset U$$

so $\rho_{\mathcal{U}}(x) \geq t > s$. That is, $\mathbf{U}^x(u - t) \subset G$ so G is open. □

Theorem 3.1. Suppose X is compact and $\mathcal{U}$ is an open covering of X. Then $l_{\mathcal{U}} > 0$.

Proof. Let

$$\mathcal{W} = \{\{x \in X : \rho_{\mathcal{U}}(x) > s\} : 0 < s < \infty\}.$$

From the Lemma we infer that $\mathcal{W}$ is an open covering of X. Since X is compact there is a finite subfamily of $\mathcal{W}$ whose union contains X. Since $\mathcal{W}$ is nested we infer that some member of $\mathcal{W}$ contains X; that is, there is s such that $0 < s < \infty$ and $X \subset \{x \in X : \rho_{\mathcal{U}}(x) > s\}$; we have $s \leq l_{\mathcal{U}}$ for any such s. □

4. Uniform continuity.

Definition 4.1. Suppose (Y, σ) is a metric space, $A \subset X$ and $f : A \to Y$. We say f is **uniformly continuous** if for each $\epsilon > 0$ there is $\delta > 0$ such that

$$a, x \in A \text{ and } \rho(x, a) < \delta \;\Rightarrow\; \sigma(f(x), f(a)) < \epsilon.$$

Theorem 4.1. Suppose (X, ρ) and (Y, σ) are metric spaces, X is compact,

$$f : X \to Y$$

and f is continuous.

Then f is uniformly continuous.

Proof. Let $\epsilon > 0$. Let

$$\mathcal{U} = \{U : U \text{ is an open subset of } X \text{ and } \mathbf{diam}\, f[U] < \epsilon\}.$$

Suppose $a \in X$. Since f is continuous at a we may choose $\eta > 0$ such that $f[\mathbf{U}^a(\eta)] \subset \mathbf{U}^{f(a)}(\epsilon/2)$. Thus, with $U = \mathbf{U}^a(\eta)$, $\mathbf{diam}\, f[U] < \epsilon$ so $\mathcal{U}$ is an open covering of X.

Since X is compact $l_{\mathcal{U}}$ is positive so we may choose δ such that $0 < \delta < l_{\mathcal{U}}$.

Suppose $a \in X$. There is $U \in \mathcal{U}$ such that $\mathbf{U}^a(\delta) \subset U$. Thus

$$\mathbf{diam}\, f[\mathbf{U}^a(\delta)] \leq \mathbf{diam}\, f[U] < \epsilon.$$

Thus

$$x \in A \text{ and } \rho(x, a) < \delta \;\Rightarrow\; \sigma(f(x), f(a)) \leq \mathbf{diam}\, f[U] < \epsilon.$$

□

5. Total boundedness.

Definition 5.1. X is **totally bounded** if for each $\epsilon > 0$ there is a finite family $\mathcal{F}$ of subsets of X such that

$$X = \bigcup \mathcal{F} \tag{1}$$

and

$$\mathbf{diam}\, F \leq \epsilon \quad \text{whenever } F \in \mathcal{F}. \tag{2}$$

Theorem 5.1. X is compact if and only if it is complete and totally bounded.

Proof. We leave as an exercise for the reader the straightforward verification that if X is compact then X is complete and totally bounded.

Suppose X is complete and totally bounded and let $\mathcal{U}$ be an open covering of X. Call a subset A of X **good** if there is a finite subfamily of $\mathcal{U}$ whose union contains A and call a subset A of X **bad** if it is not good. Note that the union of a finite family of good sets is good. We want show that X is good.

Suppose X were bad. Let $r_1, r_2, \ldots$ be a sequence of positive real numbers with limit zero. For each $i = 1, 2, \ldots$ let F_i be a finite subset of X such that

$$X = \cup\{\mathbf{B}^x(r_i) : x \in F_i\};$$

such sets exist because X is totally bounded. There would be $x_1 \in F_1$ such that $\mathbf{B}^{x_1}(r_1)$ is bad; otherwise X would be the union of the finite family $\{\mathbf{B}^x(r_1) : x \in F_1\}$ of good sets and would therefore be good. There would be $x_2 \in F_2$ such that $\mathbf{B}^{x_1}(r_1) \cap \mathbf{B}^{x_2}(r_2)$ is bad; otherwise $\mathbf{B}^{x_1}(r_1)$ would be the union of the family $\{\mathbf{B}^{x_1}(r_1) \cap \mathbf{B}^x(r_2) : x \in F_2\}$ of good sets and would therefore be good. Continuing in this way we would obtain a sequence $x_1, x_2, \ldots$ in X such that

$$C_m = \cap_{i=1}^m \mathbf{B}_{r_i}(x_i) \quad \text{would be bad.}$$

These sets would be nonempty since the empty set is good. By the completeness of X there would be a point $c \in \bigcap_{m=1}^\infty C_m$. But $c \in U$ for some $U \in \mathcal{U}$ and, since $\mathbf{diam}\, C_m$ tends to zero as m tends to infinity, we would have $C_m \subset U$ for sufficiently large m. For these m, C_m would be good. □

Corollary 5.1. A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. Suppose A is a compact subset of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is Hausdorff, A is closed by virtue of a previous Theorem. Moreover, $\{\mathbf{U}^0(r) : 0 < r < \infty\}$ is an open covering of $\mathbb{R}^n$ and therefore A; since A is compact, it has a finite subfamily whose union contains A. It follows that $A \subset \mathbf{U}^0(r)$ for some positive real number r so A is bounded.

Suppose A is a closed and bounded subset of $\mathbb{R}^n$. It follows easily from the fact that $\mathbb{R}^n$ is complete and A is closed that A , considered as a metric space, is complete. A is totally bounded as well; in fact, for any positive real number ϵ the set A is contained in the union of a finite subfamily of the family

$$\{\mathbf{B}^{\epsilon z}(\sqrt{n}\epsilon) : z \in \mathbb{Z}^n\}$$

because A is bounded. It now follows from the previous Theorem that A is compact. □

6. Lipschitz constants.

Suppose (Y, σ) is a metric space, $A \subset X$ and

$$f : A \to Y.$$

Proposition 6.1. f is continuous if and only if for each $a \in X$ and each $\epsilon \in (0, \infty)$ there is $\delta \in (0, \infty)$ such that

$$f[\mathbf{U}^a(\delta)] \subset \mathbf{U}^{f(a)}(\epsilon).$$

Proof. Proceed as we did in the case when $X = \mathbb{R}^n$ and $Y = \mathbb{R}m$ □

Definition 6.1. We let

$$\mathbf{Lip}(f)$$

be the infimum of the set of $M \in [0, \infty)$ such that

$$\sigma(f(x), f(a)) \le M\rho(x, a) \text{ whenever } x, a \in X. \tag{1}$$

note that (1) holds with $M = \mathbf{Lip}(f)$. We call this extended real number the **Lipschitz constant of** f. We say f is **Lipschitzian** if $\mathbf{Lip}(f) < \infty$. We say f is **locally Lipschitzian** if $\mathbf{Lip}(f|B) < \infty$ whenever B is a bounded subset of X.

Note that

$$\mathbf{diam}\, f[B] \le \mathbf{Lip}(f)\mathbf{diam}\, B \quad \text{whenever } B \subset X. \tag{2}$$

Proposition 6.2. If f is locally Lipschitzian then f is continuous.

Proof. This follows directly from (2). □

Theorem 6.1. Suppose Y is complete and $\mathbf{Lip}(f) < \infty$. Then f has a unique continuous extension to $\mathbf{cl}\, A$ and the Lipschitz constant of this extension equals the Lipschitz constant of f.

Proof. Let F be the set of $(a, b) \in (\mathbf{cl}\, A) \times Y$ such that

$$b \in \bigcap_{0<\delta<\infty} \mathbf{cl}\, f[\mathbf{U}^a(\delta)].$$

Since

$$f[\mathbf{U}^a(\delta)] \ne \emptyset$$

and

$$\mathbf{diam}\, f[\mathbf{U}^a(\delta)] \le \mathbf{Lip}(f)\mathbf{diam}\, \mathbf{U}^a(\delta) \le 2\delta \quad \text{for any } a \in \mathbf{cl}\, A$$

and since Y is complete we infer that F is a function whose domain is the closure of A. Since

$$f(a) \in \bigcap_{0<\delta<\infty} \mathbf{cl}\, f[\mathbf{U}^a(\delta)]$$

for any $a \in A$ we find that $F|A = f$.

Suppose $c_i \in \mathbf{cl}\, A$, $i = 1, 2$, and let r and s be positive real numbers. Since $F(c_i) \in \mathbf{cl}\, f[\mathbf{U}^{c_i}(r)]$ we may choose $a_i \in \mathbf{U}^{c_i}(r)$ such that $\sigma(F(c_i), f(a_i) < s$, $i = 1, 2$. Then

$$\begin{aligned}
\sigma(F(c_1), F(c_2)) &\le \sigma(F(c_1), f(a_1)) + \sigma(f(a_1), f(a_2)) + \sigma(f(a_2), F(c_2)) \\
&\le s + \mathbf{Lip}(f)\rho(a_1, a_2) + s \\
&\le s + \mathbf{Lip}(f)(\rho(a_1, c_1) + \rho(c_1, c_2) + \rho(c_2, a_2)) + s \\
&= 2s + \mathbf{Lip}(f)(2r + \rho(c_1, c_2));
\end{aligned}$$

owing to the arbitrariness of r and s we infer that

$$\sigma(F(c_1), F(c_2)) \le \mathbf{Lip}(f)\rho(c_1, c_2).$$

it follows that $\mathbf{Lip}(F) \le \mathbf{Lip}(f)$.

Finally, suppose that $g : \mathbf{cl}\, A \to Y$ is continuous, $g|A = f$ and $c \in \mathbf{cl}\, A$. Let $\epsilon > 0$. Then there is $\delta > 0$ such that

$$x \in \mathbf{B}^c(\delta) \cap \mathbf{cl}\, A \;\Rightarrow\; g(x) \in \mathbf{B}^{g(c)}(\epsilon).$$

Let $a \in A \cap \mathbf{B}^c(\min\{\delta, \epsilon\})$; such an a exists because $c \in \mathbf{cl}\, A$. Then, since $g(a) = f(a) = F(a)$ we have

$$\sigma(g(c), F(c)) \le \sigma(g(c), g(a)) + \sigma(F(a), F(c)) \le \epsilon + \mathbf{Lip}(F)\epsilon.$$

Owing to the arbitrariness of ϵ we infer that $g(c) = F(c)$. Thus $g = F$. $\square$