Matrices.

0.1. Definition. Whenever m and n are positive integers we let

$$\mathbf{M}_n^m = \mathbf{R}^{\{1,\ldots,m\}\times\{1,\ldots,n\}}$$

and note that $\mathbf{M}_n^m$ is a vector space. If $A \in \mathbf{M}_n^m$ we frequently depict A as a rectangular array with m rows and n columns:

$$\begin{bmatrix} A(1,1) & A(1,2) & \cdots & A(1,n) \\ A(2,1) & A(2,2) & \cdots & A(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ A(m,1) & A(m,2) & \cdots & A(m,n) \end{bmatrix}.$$

We frequently write $A_{i,j}$ for $A(i,j)$. It is a common practice to write A_{ij} for $A(i,j)$; we shall avoid this because it is clearly ambiguous. We define maps

$$\underline{\mathrm{l}} : \mathbf{M}_n^m \to \underline{\mathrm{L}}(\mathbf{R}^n;\mathbf{R}^m) \quad \text{and} \quad \underline{\mathrm{m}} : \underline{\mathrm{L}}(\mathbf{R}^n;\mathbf{R}^m) \to \mathbf{M}_n^m$$

by setting

$$\underline{\mathrm{l}}(A) = \sum_{i=1}^m \sum_{j=1}^n A(i,j)\underline{\mathrm{e}}^j \underline{\mathrm{e}}_i, \quad A \in \mathbf{M}_n^m,$$

and, for each $L \in \underline{\mathrm{L}}(\mathbf{R}^n, \mathbf{R}^m)$, letting

$$\underline{\mathrm{m}}(L)(i,j) = \underline{\mathrm{e}}^i(L(\underline{\mathrm{e}}_j)),\ (i,j) \in \{1,\ldots,m\}\times\{1,\ldots,n\}.$$

It is a simple matter to verify that these maps are linear isomorphisms which are inverse to one another.

In case $m = n$ we define the **identity matrix** I by setting

$$I = \underline{\mathrm{m}}(\underline{\mathrm{i}}_{\mathbf{R}^n})$$

and verify that

$$I(i,j) = \delta_{i,j},\ i,j = 1,\ldots,n,$$

where we have set

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

0.2. Example. Suppose $a \in \mathbf{R}^3 \sim \{\underline{0}\}$ and let L be reflection of $\mathbf{R}^3$ across $P = \{x \in \mathbf{R}^3 : x \bullet a = 0\}$. We have

$$L(x) = x - 2\frac{x \bullet a}{|a|^2}a, \quad x \in \mathbf{R}^3.$$

In particular,

$$L(\underline{\mathrm{e}}_j) = \underline{\mathrm{e}}_j - 2\frac{a_j}{|a|^2}a,\ j = 1,2,3,$$

so

$$M(i,j) = \underline{\mathrm{e}}^i(L(\underline{\mathrm{e}}_j)) = \delta_{i,j} - 2\frac{a_i a_j}{|a|^2},\ i,j = 1,2,3.$$

Thus

$$M = \frac{1}{|a|^2}\begin{bmatrix} |a|^2 - 2a_1a_1 & -2a_1a_2 & -2a_1a_3 \\ -2a_2a_1 & |a|^2 - 2a_2a_2 & -2a_2a_3 \\ -2a_3a_1 & -2a_3a_2 & |a|^2 - 2a_3a_3 \end{bmatrix}.$$

0.3. Definition. Matrix Multiplication. Suppose l, m, n are positive integers. We define **matrix multiplication** to be the map from $\mathbf{M}^l_m \times \mathbf{M}^m_n$ to $\mathbf{M}^l_n$ which assigns

$$\underline{\mathrm{m}}(\underline{\mathrm{l}}(A) \circ \underline{\mathrm{l}}(B)) \in \mathbf{M}^l_n \quad \text{to} \quad (A, B) \in \mathbf{M}^l_m \times \mathbf{M}^m_n.$$

As a consequence of these definitions we find that the algebraic properties of composition of linear maps, like associativity, carry over to multiplication of matrices.

0.4. Proposition. Suppose $A \in \mathbf{M}^m_n$ and $B \in \mathbf{M}^n_p$. Then

$$A \cdot B(i, j) = \sum_{k=1}^{n} A(i, k)B(k, j), \ i = 1, \ldots, m, \ j = 1, \ldots, p.$$

Proof. This is a straightforward calculation which we leave to the reader. □

0.5. Definition. The Inverse of a Matrix. We say $A \in \mathbf{M}^m_n$ is **invertible** if $\underline{\mathrm{l}}(A)$ is invertible in which case we set

$$A^{-1} = \underline{\mathrm{m}}(\underline{\mathrm{l}}(A)^{-1}).$$

Note that if A is invertible then $m = n$.

0.6. Definition. Suppose V is a vector space and $n = \mathbf{dim}\, V < \infty$. By an **ordered basis for** V we mean a mapping from $\{1, \ldots, n\}$ into V whose range is a basis for V. Whenever v is an ordered basis for V and $i \in \{1, \ldots, n\}$ we will frequently write v_i for $v(i)$. We will frequently write v^i for ${v_i}^*$, $i = 1, \ldots, n$. Recall that

$$v^i(v_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else,} \end{cases} \qquad i, j \in \{1, \ldots, n\}.$$

We will frequently write v^i for $v^*(i)$.

$$x = \sum_{i=1}^{n} v^i(x)v_i, \ x \in V \quad \text{and} \quad \omega = \sum_{i=1}^{n} \omega(v_i)v^i, \ \omega \in V^*.$$

We let

$$\underline{\mathrm{i}}_v :^R n \to V$$

be the linear map which carries v_j to $\underline{\mathrm{e}}_j$, $j = 1, \ldots, n$; evidently, $\underline{\mathrm{i}}_v$ carries V isomorpically onto ${}^R n$.

0.7. Definition. The Matrix of an Abstract Linear Transformation. Suppose V and W are vector spaces of dimension n and m and v and w are ordered basis for V and W, respectively. For each $L \in \underline{\mathrm{L}}(V; W)$ we define

$$\underline{\mathrm{m}}(w, L, v) \in \mathbf{M}^m_n,$$

the matrix of L **with respect to** v **and** w, by setting

$$\underline{\mathrm{m}}(w, L, v)(i, j) = \underline{\mathrm{m}}(\underline{\mathrm{i}}_w^{-1} \circ L \circ \underline{\mathrm{i}}_v).$$

It is a simple matter to verify that $\underline{\mathrm{L}}(V; W) \ni L \mapsto \underline{\mathrm{m}}(w, L, v) \in \mathbf{M}^m_n$ is a linear isomorphism as it is obviously linear and univalent. It is evident that

$$\underline{\mathrm{m}}(w, L, v)(i, j) = w^i(L(v_j)), \ i = 1, \ldots, m, \ j = 1, \ldots, n.$$

0.8. Proposition. Suppose V is a finite dimensional vector space with ordered basis v. Then

$$\underline{\mathrm{m}}(w, \underline{\mathrm{i}}_V, v) = I \;\Leftrightarrow\; v = w.$$

Proof. Immediate consequence of the definition. □

0.9. Proposition. Suppose U, V and W are vector spaces of dimensions l, m and n with ordered bases u, v and w, respectively. Then

$$\underline{\mathrm{m}}(w, M \circ L, u) = \underline{\mathrm{m}}(w, M, v) \cdot \underline{\mathrm{m}}(v, L, u), \quad L \in \underline{\mathrm{L}}(U; V),\; M \in \underline{\mathrm{L}}(V; W).$$

Proof. Exercise for the reader. □

0.10. Definition. Transposes. Suppose $A \in \mathbf{M}^m_n$. We define

$$A^t \in \mathbf{M}^n_m$$

by setting

$$A^t = \underline{\mathrm{m}}(\underline{\mathrm{e}}^*, \underline{\mathrm{l}}(A)^*, \underline{\mathrm{e}}^*)$$

where $\underline{\mathrm{e}}^*$ on the left of $\underline{\mathrm{l}}(A)^*$ is the basis of $\mathbf{R}_n$ dual to the standard basis of $\mathbb{R}n$ and $\underline{\mathrm{e}}^*$ to the left of $\underline{\mathrm{l}}(A)^*$ is the basis of $\mathbf{R}_m$ dual to the standard basis of $\mathbb{R}m$.

0.11. Proposition. Suppose $A \in \mathbf{M}^m_n$. Then

$$A^t(i, j) = A(j, i),\; i = 1, \ldots, m,\; j = 1, \ldots, n.$$

Proof. Straightforward exercise for the reader.

0.12. Proposition. Suppose V and W are finite dimensional vector spaces with ordered bases v and w, respectively, and $L \in \underline{\mathrm{L}}(V; W)$. Then

$$\underline{\mathrm{m}}(v^*, L^*, w^*) = \underline{\mathrm{m}}(w, L, v)^t.$$

Proof. Straightforward exercise for the reader. □

0.13. Proposition. Suppose V and W are finite dimensional vector spaces with ordered orthonormal bases v and w, respectively, and $L \in \underline{\mathrm{L}}(V; W)$. then

$$\underline{\mathrm{m}}(v, L^*, w) = \underline{\mathrm{m}}(w, L, v)^t,$$

where L^* here is the adjoint in the sense of innerproducts.

Proof. Straightforward exercise for the reader. □

0.14. Exercise. Let v and w be the ordered bases of $\mathbf{R}^3$ such that

$$v_1 = (1, 0, 1),\; v_2 = (3, 1, 0),\; v_3 = (2, 1, 2)$$

and

$$w_1 = (0, 0, 3),\; w_2 = (1, 1, 2),\; w_3 = (4, 0, 0).$$

(i) Determine $\underline{\mathrm{m}}(\underline{\mathrm{e}}, \underline{\mathrm{i}}_{\mathbf{R}^3}, v)$ and $\underline{\mathrm{m}}(\underline{\mathrm{e}}, \underline{\mathrm{i}}_{\mathbf{R}^3}, w)$.
Use the preceding results, and *only* the preceding results, to do the following:
(ii) Write formulae for v^1, v^2, v^3 and w^1, w^2, w^3.
(iii) Express v_1, v_2, v_3 in terms of w_1, w_2, w_3 and w_1, w_2, w_3 in terms of v_1, v_2, v_3.
(iv) Express v^1, v^2, v^3 in terms of w^1, w^2, w^3 and w^1, w^2, w^3 in terms of v^1, v^2, v^3.

Be clever about this by using the fact that the adjoint of the identity map of a vector space is the identity map of its dual space.