Inversion.

Let

$$\iota : \underset{\sim}{C} \sim \{0\} \to \underset{\sim}{C} \sim \{0\}$$

be **inversion**; that is

$$\iota(z) = \frac{1}{z} \quad \text{whenever } z \in \underset{\sim}{C} \sim \{0\}.$$

Let's give two proofs that

$$\iota'(z) = -\frac{1}{z^2}, \quad z \in \underset{\sim}{C} \sim \{0\}.$$

Suppose $a \in \underset{\sim}{C} \sim \{0\}$. Let

$$g(h) = \frac{1}{a+h} - \frac{1}{a} - \Big(-\frac{h}{a^2} \Big) \quad \text{for } h \in \underset{\sim}{C} \sim \{-a\}.$$

Proof One. Let $\epsilon > 0$ and We have

$$\begin{aligned}
|g(h)| &= \Big| \frac{a^2 - a(a+h) + h(a+h))}{a^2(a+h)} \Big| \\
&= \Big| \frac{h^2}{a^2(a+h)} \Big| \\
&\le 2\frac{|h|^2}{|a|^3} \quad \text{provided } |h| \le \tfrac{|a|}{2} \\
&\le \epsilon |h| \quad \text{provided } |h| \le \tfrac{\epsilon |a|^3}{2}.
\end{aligned}$$

So

$$g(h) \le \epsilon |h| \quad \text{if} \quad 0 < |h| \le \epsilon = \min\{\frac{|a|}{2}, \epsilon \frac{|a|^3}{2}\}.$$

Proof Two. We have

$$\frac{1}{a+h} = \frac{1}{a} \frac{1}{1 - \big(-\frac{h}{a}\big)} = \frac{1}{a} \sum_{m=0}^{\infty} \big(-\frac{h}{a} \big)^m$$

whenever $a + h \in D = \{w \in \underset{\sim}{C} : |w - a| < |a|\}$. Thus

$$g(h) = \frac{h^2}{a^3} \sum_{m=2}^{\infty} \big(-\frac{h}{a} \big)^m.$$

Our assertion now follows from our theory of infinite series.

The second proof is amenable to generalization as follows. Let X be a Banach space and let $\mathcal{I}$ be the set of invertible members of $\underset{\sim}{B}(X;X)$. Thus $A \in \mathcal{I}$ if $A \in \underset{\sim}{B}(X;X)$ and there is $B \in \underset{\sim}{B}(X,X)$ such that AB and BA equal the identity map of X. One easily verifies that such a B is unique and we denote it by

$$A^{-1}.$$

Let

$$\iota(A) = A^{-1} \quad \text{whenever } A \in \mathcal{I}.$$

As an exercise show that $\mathcal{I}$ is open and that ι is differentiable at A and determine its differential at A. Use the above to try to guess what the differential at A is.

Here is a big hint as to how to proceed. Suppose $A \in \mathcal{I}$ and $H \in \underline{\mathrm{B}}(X;X)$ is such that

$$||H|| < \frac{1}{||A||}. \tag{1}$$

If $A + H \in \mathcal{I}$ we have

$$(A+H)^{-1} = A^{-1}\big(1 - (-H \circ A^{-1})\big) = A^{-1}\sum_{m=0}^{\infty}(-H \circ A^{-1})^m.$$

Justify this; that is, show that that if (1) holds then the series converges absolutely. Then show its sum is $(A+H)^{-1}$. Finally, show that ι is differentiable at A and determine what it's differential is.

Bonus question. Show that ι has derivatives of all orders.