Integration of a differential form over an oriented submanifold.

Let m and n be positive integers, let V be an open subset of V and let $M \in \mathbf{M}_m(V)$ and let $\mathbf{o}$ be an orientation for M. We want to define a linear mapping

$$\int_M \cdot : \mathcal{A}_0^m(V) \to \mathbf{R}$$

which "does the right thing"; what the "right thing" is will be spelled out below.

Lemma. Suppose $\omega \in \mathcal{A}_0^m(V)$,

$$(U_i, \phi_i) \in \mathcal{P}(M, V),\ i = 1, 2$$

and

$$\mathbf{spt}\,\omega \subset U_1 \cap U_2.$$

Then

$$\mathbf{o}(U_1, \phi_1) \int_{\phi_1^{-1}[M]} {\phi_1}^{\#}\omega(t_1)(\mathbf{e}_1, \ldots, \mathbf{e}_m)\, dt_1 = \mathbf{o}(U_2, \phi_2) \int_{\phi_2^{-1}[M]} {\phi_2}^{\#}\omega(t_2)(\mathbf{e}_1, \ldots, \mathbf{e}_m)\, dt_2.$$

Proof. Because $\mathbf{spt}\,\omega \subset U_1 \cap U_2$ we find that

$$\mathbf{o}(U_1, \phi_1) \int_{\phi_1^{-1}[M]} {\phi_1}^{\#}\omega(t)(\mathbf{e}_1, \ldots, \mathbf{e}_m)\, dt = \mathbf{o}(U_1, \phi_1) \int_{\phi_1^{-1}[U_2 \cap M]} {\phi_1}^{\#}\omega(t)(\mathbf{e}_1, \ldots, \mathbf{e}_m)\, dt$$

and

$$\mathbf{o}(U_2, \phi_2) \int_{\phi_2^{-1}[M]} {\phi_2}^{\#}\omega(t)(\mathbf{e}_1, \ldots, \mathbf{e}_m)\, dt = \mathbf{o}(U_2, \phi_2) \int_{\phi_2^{-1}[U_1 \cap M]} {\phi_2}^{\#}\omega(t)(\mathbf{e}_1, \ldots, \mathbf{e}_m)\, dt.$$

From the Change of Variables Formula for Multiple Integrals and the fact that $(\phi_1^{-1}[U_2 \cap M], \phi_2^{-1} \circ \phi_1 | \phi_1^{-1}[U_2 \cap M], \phi_2^{-1}[U_1 \cap M]) \in \mathbf{Diffeo}_m$ we infer that

$$\begin{aligned}
&\mathbf{o}(U_2, \phi_2) \int_{\phi_2^{-1}[U_1 \cap M]} {\phi_2}^{\#}\omega(t_2)(\mathbf{e}_1, \ldots, \mathbf{e}_m)\, dt_2 \\
&\qquad = \mathbf{o}(U_2, \phi_2) \int_{\phi_1^{-1}[U_2 \cap M]} {\phi_2}^{\#}\omega(\phi_2^{-1} \circ \phi_1(t_1))(\mathbf{e}_1, \ldots, \mathbf{e}_m)\, |\mathbf{det}\, \partial(\phi_2^{-1} \circ \phi_1)(t_1)|\, dt_1.
\end{aligned}$$

Suppose $t_i \in T_i$ and $\phi_1(t_1) = \phi_2(t_2)$. The Lemma will follow if we can show that

$$\mathbf{o}(U_2, \phi_2){\phi_2}^{\#}\omega(\phi_2^{-1} \circ \phi_1(t_1))(\mathbf{e}_1, \ldots, \mathbf{e}_m)\, |\mathbf{det}\, \partial(\phi_2^{-1} \circ \phi_1)(t_1)| = \mathbf{o}(U_1, \phi_1){\phi_1}^{\#}\omega(t_1)(\mathbf{e}_1, \ldots, \mathbf{e}_m).$$

Then

$$\mathbf{o}(U_1, \phi_1) = \mathbf{sign}\ \mathbf{det}\, \partial(\phi_2^{-1} \circ \phi_1)(t_1)\, \mathbf{o}(U_2, \phi_2)$$

and

$$\begin{aligned}
\phi_1^{\#}\omega(t_1)(\mathbf{e}_1, \ldots, \mathbf{e}_m) &= \big(\phi_2 \circ (\phi_2^{-1} \circ \phi_1)\big)^{\#}\omega(t_1)(\mathbf{e}_1, \ldots, \mathbf{e}_m) \\
&= \big((\phi_2^{-1} \circ \phi_1)^{\#}\phi_2^{\#}\omega\big)(t_1)(\mathbf{e}_1, \ldots, \mathbf{e}_m) \\
&= \mathbf{det}\, \partial(\phi_2^{-1} \circ \phi_1)(t_1)\, \phi_2^{\#}\omega(t_2)(\mathbf{e}_1, \ldots, \mathbf{e}_m).
\end{aligned}$$

□

Theorem. There is one and only one linear function

$$\int_M \cdot : \mathcal{A}_0^m(V) \to \mathbf{R}$$

such that

$$\int_M \omega = \mathbf{o}(U,\phi) \int_{\phi^{-1}[M]} \phi^{\#}\omega(t)(\mathbf{e},\ldots,\mathbf{e}_m)\,dt$$

whenever $(U,\phi) \in \mathcal{P}(M,V)$, $\omega \in \mathcal{A}_0^m(V)$ and $\mathbf{spt}\,\omega \subset U$. Moreover,

$$\int_M \omega = \sum_{(U,\phi,\chi)\in\mathcal{A}} \int_{\phi^{-1}[M]} \phi^{\#}(\chi\omega)(t)(\mathbf{e}_1,\ldots,\mathbf{e}_m)\,dt$$

whenever $\omega \in \mathcal{A}_0^m(V)$ and $\mathcal{A}$ is an admissible subfamily of $\mathcal{Q}(M,V)$.

Proof. For each admissible subfamily $\mathcal{A}$ of $\mathcal{Q}(M,V)$ we set

$$J_{\mathcal{A}}(\omega) = \sum_{(U,\phi,\chi)\in\mathcal{A}} \int_{\phi^{-1}[M]} \phi^{\#}(\chi\omega)(t)(\mathbf{e}_1,\ldots,\mathbf{e}_m)\,dt \quad \text{whenever } \omega \in \mathcal{A}_0^m(V).$$

Suppose $(\tilde{U},\tilde{\phi},\tilde{\chi}) \in \mathcal{Q}(M,V)$ and $\mathcal{A}$ is an admissible subfamily of $\mathcal{Q}(M,V)$. Then

$$\begin{aligned}
&\sum_{(U,\phi,\chi)\in\mathcal{A}} \tilde{\phi}^{\#}(\chi\omega)(\tilde{t})(\mathbf{e}_1,\ldots,\mathbf{e}_m)\,d\tilde{t} \\
&\qquad = \tilde{\phi}^{\#}\big(\sum_{(U,\phi,\chi)\in\mathcal{A}} (\chi\omega)\big)(\tilde{t})(\mathbf{e}_1,\ldots,\mathbf{e}_m) \\
&\qquad = \tilde{\phi}^{\#}\big(\big(\sum_{(U,\phi,\chi)\in\mathcal{A}} \chi\big)\omega\big)(\tilde{t})(\mathbf{e}_1,\ldots,\mathbf{e}_m) \\
&\qquad = \tilde{\phi}^{\#}\omega(\tilde{t})(\mathbf{e}_1,\ldots,\mathbf{e}_m)
\end{aligned}$$

for any $\tilde{t} \in \mathbf{U}^m$. Then, by the preceding Lemma, we have for any $\omega \in \mathcal{D}(V)$ such that $\mathbf{spt}\,\omega \subset \tilde{U}$ that

$$\begin{aligned}
J_{\mathcal{A}}(\omega) &= \sum_{(U,\phi,\chi)\in\mathcal{A}} \mathbf{o}(U,\phi) \int_{\phi^{-1}[M]} \phi^{\#}(\chi\omega)(t)(\mathbf{e}_1,\ldots,\mathbf{e}_m)\,dt \\
&= \sum_{(U,\phi,\chi)\in\mathcal{A}} \mathbf{o}(\tilde{U},\tilde{\phi}) \int_{\tilde{\phi}^{-1}[M]} \tilde{\phi}^{\#}(\chi\omega)(\tilde{t})(\mathbf{e}_1,\ldots,\mathbf{e}_m)\,d\tilde{t} \\
&= \mathbf{o}(\tilde{U},\tilde{\phi}) \int_{\tilde{\phi}^{-1}[M]} \sum_{(U,\phi,\chi)\in\mathcal{A}} \tilde{\phi}^{\#}(\chi\omega)(\tilde{t})(\mathbf{e}_1,\ldots,\mathbf{e}_m)\,d\tilde{t} \\
&= \mathbf{o}(\tilde{U},\tilde{\phi}) \int_{\tilde{\phi}^{-1}[M]} \tilde{\phi}^{\#}\omega(\tilde{t})(\mathbf{e}_1,\ldots,\mathbf{e}_m)\,d\tilde{t}.
\end{aligned} \tag{1}$$

Suppose $\mathcal{A}_i,\ i = 1,2$, are admissible subfamilies of $\mathcal{Q}(M,V)$. The proof will be complete if can show that

$$J_{\mathcal{A}_1} = J_{\mathcal{A}_2}.$$

But for any $\omega \in \mathcal{A}_0^m(V)$ we may use (1) to calculate

$$\begin{aligned}
J_{\mathcal{A}_1}(\omega) &= \sum_{(U_2,\phi_2,\chi_2)\in\mathcal{A}_2} J_{\mathcal{A}_1}(\chi_2\omega) \\
&= \sum_{(U_2,\phi_2,\chi_2)\in\mathcal{A}_2} \int_{\phi_2^{-1}[M]} {\phi_2}^{\#}(\chi_2\omega)(t_2)(\mathbf{e}_1,\ldots,\mathbf{e}_m)\,dt_2 \\
&= J_{\mathcal{A}_2}(f).
\end{aligned}$$