1. Introduction to the theory of infinite sets.

Theorem 1.1. $\mathbb{N}$ is infinite.

Proof. We have $\mathbb{N} \approx \mathbb{N} \sim \{0\}$ by means of S. Thus $\mathbb{N}$ cannot be finite since, as we have already shown, no finite set is equipotent with a proper subset. □

Theorem 1.2. Suppose $A \subset \mathbb{N}$ and A is infinite. Then $A \approx \mathbb{N}$.

Proof. We do this by defining a function by induction. Let $\mathcal{G} = \{g : \text{for some } n \in \mathbb{N}, g : \mathbb{I}(n) \to A\}$ and define $G : \mathcal{G} \to A$ by letting $G(g)$ be the least element of $A \sim \operatorname{rng} g$ for $g \in \mathcal{G}$. One obtains a function $f : \mathbb{N} \to A$ such that $f(n) = G(f|\mathbb{I}(n))$ for each $n \in \mathbb{N}$. We leave to the reader the straightforward verification that f is univalent with range equal to A. □

Remark 1.1. Note that in the above proof f carries n to the n-th least element of A. Alternatively, we could use the ordering on $\mathbf{N}$ to induce a well ordering on A and then use our previous results about well ordered sets.

Theorem 1.3. $\mathbb{N}^n \approx \mathbb{N}$ for any $n \in \mathbb{N}^+$.

Proof. The statement is trivially true when $n = 1$ and will follow by induction on n if we can show it holds for $n = 2$.

We now show that $\mathbb{N} \approx \mathbb{N} \times \mathbb{N}$. For each $n \in \mathbf{N}$ we let $L(n) = \{(l, m) \in \mathbb{N} \times \mathbb{N} : l + m = n\}$ and we let $M(n) = \bigcup\{L(m) : m < n\}$. Note that $L(n)$ is finite, $|L(n)| = n + 1$ and that $\{L(n) : n \in \mathbb{N}\}$ is a partition of $\mathbb{N} \times \mathbb{N}$. We define $f : \mathbb{N} \to \mathbb{N} \times \mathbf{N}$ by requiring that $f(|M(k)| + n) = (l, m)$ whenever $0 \le n < k + 1$ and $l + m = n$. We leave it to the reader to verify that f is univalent with range $\mathbb{N} \times \mathbb{N}$. □

Theorem 1.4. A set A is countable if and only if there is a function with domain $\mathbb{N}$ and range A.

Proof. If A is countable then the existence of such a function is clear.

Suppose $f : \mathbb{N} \to A$ and $\mathbf{rng}\, f = A$. Let B be the set of those $n \in \mathbb{N}$ such that, for some $a \in A$, n is the least element of $f^{-1}[\{a\}]$. Then, by earlier work, B is countable and $f|B$ is univalent with range A. □

Theorem 1.5. Suppose $\mathcal{A}$ is a countable family of countable sets. Then $\cup\mathcal{A}$ is countable.

Proof. Let $F : \mathbb{N} \to \mathcal{A}$ have range $\mathcal{A}$ For each $A \in \mathcal{A}$ let $\mathbf{n}(A)$ be the set of functions with domain $\mathbb{N}$ and range A. Let c be a choice function for the family $\{\mathbf{n}(A) : A \in \mathcal{A}\}$. Define $f : \mathbf{N} \times \mathbb{N} \to A$ to have the value $f(m, n) = c(F(m))(n)$ at $(m, n) \in \mathbb{N} \times \mathbb{N}$. Note that the range of f equals A. Thus, as A is the image of a countable set, it is countable. □

Theorem 1.6. $\{F : F \subset \mathbb{N} \text{ and } F \text{ is finite}\}$ is countable.

Proof. For $n \in \mathbb{N}$ let $\mathcal{F}_n = \{F : F \subset \mathbb{N} \text{ and } |F| = n\}$. Let $F_n : \mathbf{N} \to \mathcal{F}_n$ assign to $m \in \mathbb{N}$ the m-th member of $\mathcal{F}_n$ in the dictionary order. Let $f : \mathbb{N} \times \mathbf{N} \to \bigcup \mathcal{F}_n : n \in \mathbb{N}\}$ assign $F_n(m)$ to $(m, n) \in \mathbb{N} \times \mathbb{N}$. Then f has range $\{F : F \subset \mathbb{N} \text{ and } F \text{ is finite}\}$. □

The following interesting Theorem uses our theory of natural numbers. It says that cardinal numbers are linearly ordered. You can give what is in some ways a simpler proof of this Theorem by using well ordering. The proof we give is more constructive.

Theorem 1.7. Schroeder-Bernstein. Suppose A and B are sets such that A is equipotent with a subset of B and B is equipotent with a subset of A. Then $A \approx B$.

Proof. We need the following

Lemma 1.1. Suppose C is a set and $h : C \to C$ is univalent. Let $h^0(x) = x$ for $x \in C$ and let $h^{n+1} = h \circ h^n$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $C_n = h^n[C] \sim h^{n+1}[C]$. Let $C_\infty = \cap_{n=0}^\infty h^n[C]$. Then

$$(h| \cup_{m=0}^\infty C_{2m}) \cup \ (h^{-1}| \cup_{m=0}^\infty C_{2m+1}) \cup \ (h|C_\infty)$$

is a univalent function with domain and range equal to C.

Proof. Evidently, $\{C_n : n \in \mathbb{N}\} \cup \{C_\infty\}$ is a partition of C. For $n \in \mathbb{N}$ we have $h[C_{2m}] = C_{2m+1}$ and $h^{-1}[C_{2m+1}] = C_{2m}$ as well as $h[C_\infty] = C_\infty$. □

Remark 1.2. The set C_n and the functions h^n are constructed by induction. We are also using here that for each $n \in \mathbb{N}$ there is a unique $m \in \mathbb{N}$ such that either $n = 2m$ or $n = 2m + 1$.

Proof. We leave this as a straightforward exercise for the reader. □

We may suppose that $A \cap B = \emptyset$. Let $C = A \cup B$ and let $h : C \to C$ be such that $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$. Let H be the function produced by the Lemma. Since $h[A] \subset B$, $h[B] \subset A$, $h^{-1}[A] \subset B$ and $h^{-1}[B] \subset A$ we infer that $H[A] \subset B$ and $H[B] \subset A$ so range $H|A = B$. □

Theorem 1.8.

1.1. Corollary. $2^{\mathbb{N}}$ is uncountable.

Proof. A set is countable if and only if it equals the range of a function with domain $\mathbb{N}$. Thus, were $2^{\mathbb{N}}$ countable, there would exist a function f with domain $\mathbb{N}$ and range $2^{\mathbb{N}}$ which is impossible by virtue of a preceding Theorem. □

Theorem 1.9. $2^{\mathbb{N}} \approx \mathbb{N}^{\mathbb{N}}$.

Proof. Define $f : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by requiring that, for each $X \subset \mathbb{N}$, $f(X)$ is 1 on X and 0 on $\mathbb{N} \sim X$. Note that f is univalent. Let J be a univalent function on $\mathbb{N} \times \mathbb{N}$ with range equal to $\mathbb{N}$. Define $g : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ by requiring that $g(\mu) = J[\ \mu]$ for $\mu \in \mathbb{N}^{\mathbb{N}}$. Note that g is univalent. Apply the Schroeder-Bernstein Theorem to complete the proof. □

Theorem 1.10. Suppose A is a countable subset of $2^{\mathbb{N}}$. Then $2^{\mathbb{N}} \sim A \approx 2^{\mathbb{N}}$.

Proof. We will prove the statement obtained from the theorem by replacing each occurrence of '$2^{\mathbb{N}}$' by '$\mathbb{N}^{\mathbb{N}}$'. Let $F : \mathbb{N} \to A$ be such that rng $F = A$. Define $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \sim A$ by letting $f(\alpha)(n) = \alpha(n) + F(n)(n) + 1$ for $n \in \mathbb{N}$. Note that f is univalent. Apply the Schroeder-Bernstein Theorem to complete the proof. □