Homework Seven. Due Friday, October 16

1. The Hausdorff distance.

Suppose (X, ρ) is a metric space.

Exercise 1.1. Suppose $A \subset X$ and let

$$\rho(x, A) = \inf\{\rho(x, a) : a \in A\}.$$

Show that

$$|\rho(x, A) - \rho(y, A)| \le \rho(x, y) \quad \text{whenever } x, y \in X.$$

Hint: Check the hints in the notes on metric spaces.

Exercise 1.2. (This is challenging.) Let $\mathcal{B}(X)$ be the family of closed subsets of X with finite diameter. Let

$$\rho(A, B) = \max\{\sup\{\rho(b, A) : b \in B\}, \sup\{\rho(a, B) : a \in A\}\}.$$

(Note that poor little ρ is being used in three different ways here!). Show that $\mathcal{B}(X) \times \mathcal{B}(X) \ni (A, B) \mapsto \rho(A, B) \in [0, \infty)$ is a metric on $\mathcal{B}(X)$.

Extra credit: Show that $\mathcal{B}(X)$ is compact if X is compact.

2. Oscillation.

Suppose X is a topological space, Y is a metric space and $f : X \to Y$. We define

$$\mathbf{osc}\, f : X \to [0, \infty],$$

the **oscillation of** f by setting

$$\mathbf{osc}\, f(a) = \inf\{\mathbf{diam}\, f[U] : a \in U \subset X \text{ and } U \text{ is open}\} \quad \text{for } a \in X.$$

Show that

(i) if $a \in X$ then f is continuous at a if and only if $\mathbf{osc}\, f(a) = 0$;
(ii) the set $\{x \in X : \mathbf{osc}\, f(x) \ge c\}$ is closed for any $c \in [0, \infty]$.

3. Rearranging sums.

Suppose X and Y are sets, $\sigma : X \to Y$, σ is univalent and $\mathbf{rng}\, \sigma = Y$. In both of the Exercises which follow you should make use of the stuff on Summation that has an ϵ in it.

Exercise 3.1. Suppose $p : Y \to [0, \infty]$. Show that

$$\sum_{\sigma[A]} p = \sum_A p \circ \sigma \quad \text{whenever } A \subset X.$$

Exercise 3.2. Suppose V is a Banach space, $f : Y \to V$, $A \subset X$ and $f \circ \sigma$ is summable over A. Show that f is summable over $\sigma[A]$ and that

$$\sum_{\sigma[A]} f = \sum_A f \circ \sigma.$$

4. The Fundamental Theorem of Algebra

Suppose d is a positive integer and p is a nonconstant complex polynomial function of degree d. This means, by definition, that there are complex numbers $c_0, c_1, \ldots, c_d$ such that $c_d \neq 0$ and

$$p(z) = \sum_{j=0}^{d} c_j z^j \quad \text{whenever} \quad z \in \mathbb{C}.$$

Show that there is $a \in \mathbb{C}$ such that $p(a) = 0$.

I suggest you proceed as follows.

Formulate precisely and prove that

$$\lim_{|z| \leftarrow \infty} |p(z)| = \infty. \tag{1}$$

Next, set

$$m = \inf\{|p(z)| : z \in \mathbb{C}\}$$

and show that there exists

$$a \in \mathbb{C} \quad \text{such that} \quad |p(a)| = m.$$

Do this by using (1) to obtain $R \in (0, \infty)$ such that $|p(z)| \geq m$ whenever $|z| \geq R$ and then minimizing $|p|$ on $\{z \in \mathbb{C} : |z| \leq R\}$.

Finish things off by showing that $m = 0$ so that $p(a) = 0$. Do this by showing that if $|p(a)| > 0$ then there are $s > 0$ and $t \in \mathbb{R}$ such that $|p(a + se^{it})| < |p(a)|$. For this purpose it will help to write $p(z) = p(a) + q(z)(z-a)^l$ where l is a positive integer and where q is a polynomial function such $q(a) \neq 0$. Then write $q(z) = q(a) + r(z)(z-a)^m$ where r is a polynomial function such $r(a) \neq 0$. One then has

$$p(z) = p(a) + (q(a) + (z-a)^m r(z))(z-a)^l.$$

Now let $z = a + se^{it}$ where $0 < s < \infty$ and $t \in \mathbb{R}$ to obtain

$$p(z) = p(a) + (q(a) + s^m e^{imt} r(a + se^{it})) s^l e^{ilt}.$$

Finally, choose t such that $q(a)e^{ilt} = -wp(a)$ for some *positive* real number w; *here we are using the fact that the range of the complex exponential function is* $\mathbb{C} \sim \{0\}$. Letting $s \downarrow 0$ one obtains a contradiction.