Homework Six. Due Wednesday, October 7, 2008

1. Some simple stuff with inf and sup.

Exercise 1.1. Suppose A is nonempty closed subset of $\mathbb{R}$. Show that if A has an upper bound then $\sup A \in A$. Show that if A has a lower bound then $\inf A \in A$.

2. More with compactness.

Exercise 2.1. Suppose X is a topological space, K is a compact subset of X, $f : K \to \mathbb{R}$ and f is continous. Show that there are $m, M \in K$ such that

$$f(m) \le f(x) \le f(M) \quad \text{whenever } x \in K.$$

Exercise 2.2. Let

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

Show that $\mathbb{S}^{n-1}$ is compact by showing that it is closed and bounded. (Hint: It's obvious that $\mathbb{S}^{n-1}$ is bounded. Let $f(x) = |x|$ for $x \in \mathbb{R}^n$. Show that $\mathbf{Lip}(f) = 1$. Note that $\mathbb{S}^{n-1} = f^{-1}[\{1\}]$ and that $\{1\}$ is a closed subset of $\mathbb{R}$.)

3. Convexity.

A subset C of a vector space is **convex** if

$$a, b \in C \;\Rightarrow\; \{(1-t)a + tb : 0 < t < 1\} \subset C.$$

Exercise 3.1. Suppose C is a subset of $\mathbb{R}^n$ with the following properties:

(i) $0 \in C$;
(ii) If $x \in C$ then $-x \in C$;
(iii) C is convex;
(iv) C is bounded;
(v) C is open.

We let

$$\rho(x) = \inf\left\{\frac{1}{t} : 0 \le t < \infty \text{ and } tx \in C\right\} \quad \text{for } x \in \mathbb{R}^n.$$

Show that

(i) $\rho(0) = 0$;
(ii) $0 < \rho(x) < \infty$ if $x \in \mathbb{R}^n \sim \{0\}$;
(iii) $\rho(cx) = |c|\rho(x)$ if $c \in \mathbb{R}$ and $x \in \mathbb{R}^n$;
(iv) $\rho(x+y) \le \rho(x) + \rho(y)$ if $x, y \in \mathbb{R}^n$.

That is, ρ is a norm on $\mathbb{R}^n$.

Show that

$$C = \{x \in \mathbb{R}^n : \rho(x) < 1\}.$$

Exercise 3.2. Suppose ρ is a norm on $\mathbb{R}^n$. Let

$$m = \inf\{\rho(x) : x \in \mathbb{R}^n \text{ and } |x| = 1\}$$

and let

$$M = \sum_{i=1}^{n} \rho(\mathbf{e}_i).$$

Show that

(i) $0 < M < \infty$ and
$$\rho(x) \le M|x| \quad \text{for } x \in \mathbb{R}^n;$$
(ii) ρ is continuous;
(iii) $0 < m$ and
$$m|x| \le \rho(x) \quad \text{for } x \in \mathbb{R}^n;$$
(Hint: For (iii) you will need to use the compactness of $\mathbb{S}^{n-1}$.)

Exercise 3.3. Suppose ρ_i, $i = 1, 2$, are norms on $\mathbb{R}^n$. Show that they induce the same topology on $\mathbb{R}^n$. (Hint: It will suffice to show this when ρ_2 is the standard Euclidean norm on $\mathbb{R}^n$; the assertion to be proved in this case follows rather directly from the foregoing results.)

4. Univalence.

Suppose I is an interval, $f : I \to \mathbb{R}$ and f is continuous. (I remind you that we have already shown that $f[I]$ is an interval.)

Exercise 4.1. Show that if f is univalent then f is either increasing or decreasing.

Exercise 4.2. Show that if f is univalent then f^{-1} is continuous.