1. Homework Ten. Due Monday, November 23

2. An introduction to convex functions.

1. Suppose $I \subset \mathbb{R}$ and

$$f : I \to \mathbb{R}.$$

We say f is **convex** if I is an interval and

$$f((1-\lambda)a + \lambda b) \le (1-\lambda)f(a) + \lambda f(b) \quad \text{whenever } a, b \in I \text{ and } 0 \le \lambda \le 1.$$

Prove the following.

(i) f is convex if and only if

$$\{(x,y) \in I \times \mathbb{R} : f(x) \le y\} \text{ is convex.}$$

(ii) f is convex if and only if for each $a \in I$ there is $m \in \mathbb{R}$ such that

$$f(a) + m(x-a) \le f(x) \quad \text{whenever } x \in I.$$

(iii) f is convex if and only if

$$\frac{f(b)-f(a)}{b-a} \le \frac{f(d)-f(c)}{d-c}$$

whenever $a, b, c, d \in I$, $a < b$, $c < d$, $a \le c \le b \le d$;

(iv) if $a, b \in I$ and $a < b$ there is $M \in [0, \infty)$ such that

$$|f(x) - f(y)| \le M|x-y| \quad \text{whenever } a < x \le y < b.$$

(v) if f is continuous and twice differentiable in the interior of I then f is convex if and only if f'' is nonnegative.

2. Suppose C is a compact convex subset of $\mathbb{R}^2$, $I = \{x : (x,y) \in C \text{ for some } y\}$ and $f_\pm : I \to \mathbb{R}$ are such that

$$f_+(x) = \sup\{y : (x,y) \in C\} \quad \text{and} \quad f_-(x) = \inf\{y : (x,y) \in C\}.$$

Show that I is an interval, f_- is convex, f^+ is concave and

$$C = \{(x,y) \in \mathbb{R}^2 : x \in I \text{ and } f_(x) \le y \le f_+(x)\}.$$

2.1. **The inequalities of Hölder and Minkowski; Convolution.** Prove Proposition 1.1, Theorem 1.1, Theorem 1.2, Theorem 1.3 from **Hölder's Inequality and and Minkowski's Inequality**.

Prove Proposition 2.1, Proposition 2.2, and Proposition 2.3 from **Convolution and other stuff.**