Inner products.

Let V be a vector space.

Definition 0.1. We say a function

$$\beta : V \times V \to \mathbb{R}$$

is an **inner product on** V if

(i) for each $v \in V$ the function

$$V \ni w \mapsto \beta(v, w) \in \mathbb{R}$$

is linear;

(ii) for each $w \in V$ the function

$$V \ni v \mapsto \beta(v, w) \in \mathbb{R}$$

is linear;

(iii) $\beta(v, w) = \beta(w, v)$ for each $v, w \in V$ and

(iv) $\beta(v, v) \geq 0$ with equality only if $v = 0$.

Here's some fancy mathematics terminology that goes with this. Properties (i) and (ii) say that i is **bilinear**, property (iii) says that i is **symmetric** and property (iv) says that i is **positive definite**.

One often writes

$$v \bullet w$$

for $\beta(v, w)$. Keeping in mind (iv), for $v \in V$ we set

$$|v| = \sqrt{v \bullet v}$$

and call this nonegative real number the **norm** or **length** of v.

We have the all important

Theorem 0.1 (Cauchy-Schwarz inequality.)**.** For any $v, w \in V$ we have

$$|v \bullet w| \leq |v|\,|w|,$$

with equality only if the set $\{v, w\}$ is dependent.

Proof. We shall assume the $w \neq 0$ since otherwise the assertion holds trivially. For any $t \in \mathbb{R}$ we have

$$\begin{aligned} 0 &\leq |v + t\,w|^2 \\ &= (v + t\,w) \bullet (v + t\,w) \\ &= v \bullet v + v \bullet (tw) + (tw) \bullet v + (tw) \bullet (tw) \\ &= v \bullet v + t\,v \bullet w + t\,w \bullet v + t^2\, w \bullet w \\ &= |v|^2 + 2\,t\,v \bullet w + t^2\,|w|^2. \end{aligned}$$

Setting

$$t = -\frac{v \bullet w}{|w|^2}$$

and doing a little bit of manipulation we infer the desired inequality. If $|v \bullet w| = |v||w|$ we find that $|v + tw| = 0$ so that $\{v, w\}$ is indeed dependent. □

Theorem 0.2 (The triangle inequality.)**.** For any $v, w \in V$ we have that

$$|v + w| \le |v| + |w|,$$

with equality only if $\{v, w\}$ is dependent.

Proof. Square both sides and use the Cauchy-Schwartz inequality. □

Theorem 0.3 (The Parallelogram Law.)**.** Suppose $v, w \in V$. Then

$$|v + w|^2 + |v - w|^2 = 2(|v|^2 + |w|^2).$$

Proof. Turn the crank. □

Things to do.

i) Justify the use of term *length* above.

ii) We say v **is perpendicular to** w and write

$$v \perp w$$

if

$$v \bullet w = 0.$$

Justify the use of this terminology.

iii) Suppose f and g are continuous real value functions on the interval $[a, b]$. Show that

$$|\int_a^b f(x)g(x)\,dx| \le (\int_a^b f(x)^2\,dx)^{\frac{1}{2}} (\int_a^b g(x)^2\,dx)^{\frac{1}{2}}.$$

Hilbert space

Let A be a set.

We let

$$\mathbf{H}_A = \{x \in \mathbb{R}^A : \sum_{\alpha \in A} x_\alpha^2 < \infty\}.$$

Suppose $x, y \in \mathbf{H}_A$ and F is a finite subset of A. By the Cauchy-Schwarz inequality in $\mathbb{R}^F$ we have

$$\Big(\sum_{\alpha \in F} |x_\alpha y_\alpha|\Big)^2 \le \Big(\sum_{\alpha \in F} x_\alpha^2\Big)\Big(\sum_{\alpha \in F} y_\alpha^2\Big) \le \Big(\sum_{\alpha \in \alpha} x_\alpha^2\Big)\Big(\sum_{\alpha \in A} y_\alpha^2\Big) < \infty.$$

Thus we may define

$$x \bullet y = \sum_{\alpha \in A} x_\alpha y_\alpha$$

and thereby make $\mathbf{H}_A$ into an inner product space. We have

$$|x|^2 = \sum_{\alpha \in A} x_\alpha^2 \text{ for } x \in \mathbf{H}_A.$$

For each $\alpha \in A$ we define $\mathbf{e}_\alpha \in \mathbf{H}_A$ by setting

$$(\mathbf{e}_\alpha)_\beta = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{else,} \end{cases}$$

we note the $|\mathbf{e}_\alpha| = 1$ and we define the linear map

$$\mathbf{e}^\alpha : \mathbf{H}_A \to \mathbb{R}$$

by setting $\mathbf{e}^\alpha(x) = x_\alpha$. Evidently,

$$|\mathbf{e}^\alpha(x)| \le |x|, \quad x \in \mathbf{H}_A.$$

For each subset B of A we define the linear map

$$P_B : \mathbf{H}_A \to \mathbf{H}_A$$

by setting

$$P_B(x)_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in B, \\ 0 & \text{else} \end{cases}$$

for $x \in \mathbf{H}_A$. We have

$$|P_{B\cup C}(x)|^2 + |P_{B\cap C}(x)|^2 = |P_B(x)|^2 + |P_C(x)|^2 \quad \text{whenever } B, C \subset A \text{ and } x \in \mathbf{H}_A.$$

Theorem 0.4. $\mathbf{H}_A$ is complete.

Proof. Suppose $\mathcal{C}$ is a nonempty family of nonempty closed subsets of $\mathbf{H}_A$ such that

$$\inf\{\mathbf{diam}\, C : C \in \mathcal{C}\} = 0. \tag{1}$$

For each $\alpha \in A$ let $\mathcal{C}_\alpha = \{\mathbf{cl}\, \mathbf{e}^\alpha[C] : C \in \mathcal{C}$. Inasmuch as

$$\mathbf{diam}\, \mathbf{cl}\, \mathbf{e}^\alpha[C] \le \mathbf{diam}\, C \text{ whenever } C \in \mathcal{C}$$

we infer that $\mathcal{C}_\alpha$ is a nonempty family of nonempty closed subsets of $\mathbb{R}$ such $\inf\{\mathbf{diam}\, C : C \in \mathcal{C}_\alpha\} = 0$ whenever $\alpha \in A$. Owing to the completeness of $\mathbb{R}$ we may define $x \in \mathbb{R}^A$ by requiring that

$$x_\alpha \in \bigcap\{\mathbf{cl}\, \mathbf{e}^\alpha[C] : C \in \mathcal{C}\},\ \alpha \in A.$$

Note that

$$|x_\alpha - y_\alpha| \le \mathbf{diam}\, C \text{ whenever } y \in C \in \mathcal{C}. \tag{2}$$

To prove the Theorem we will show that

$$x \in \cap\mathcal{C}. \tag{3}$$

Lemma 0.1. $x \in \underline{\mathrm{H}}_A$.

Proof. Suppose F is a finite subset of A. Using the Cauchy-Schwartz Inequality in $\underline{\mathrm{R}}^F$ we find that, for any $y \in C \in \mathcal{C}$,

$$|x|F| \le |(x-y)|F| + |y|F| = \sqrt{\sum_{\alpha\in F} (x_\alpha - y_\alpha)^2} + |P_F(y)| \le \sqrt{|F|}\mathbf{diam}\, C + |y|.$$

That $x \in \underline{\mathrm{H}}_A$ now follows from (1). $\square$

Lemma 0.2. Suppose $C \in \mathcal{C}$ and $\epsilon > 0$. There exists a finite subset G of A such that

$$|P_{A\sim G}(y)| \le \mathbf{diam}\, C + \epsilon \quad \text{for } y \in C.$$

Proof. Let $z \in C$. Let G be a finite subset of A such that $|P_{A\sim G}(z)| < \epsilon$. For any $y \in C$ we have

$$|P_{A\sim G}(y)| \le |P_{A\sim G}(y-z)| + |P_{A\sim G}(z)| \le |y-z| + |P_{A\sim G}(z)| \le \mathbf{diam}\, C + \epsilon$$

□

Suppose $C \in \mathcal{C}$ and $\epsilon > 0$. (3) will follow if we can show that

$$\text{(4)} \qquad \underline{\mathrm{B}}_x(\epsilon) \cap C \ne \emptyset.$$

Using Lemma Two and (1) choose a finite subset G of A and $D \in \mathcal{C}$ such that $D \subset C$ and $|P_{A\sim G}(y)| \le \epsilon/3$ whenever $y \in D$. Next use Lemma One to choose a finite subset F of A such that $G \subset F$ and $|P_{A\sim F}(x)| \le \epsilon/3$. Finally use (1) to choose $E \in \mathcal{C}$ such that $E \subset D$ and $\sqrt{|F|}\mathbf{diam}\, E \le \epsilon/3$. Let $y \in D$. Then by (2)

$$|x-y| \le |P_F(x-y)| + |P_{A\sim F}(x)| + |P_{A\sim F}(y)| \le \sqrt{F}\mathbf{diam}\, E + |P_{A\sim F}(x)| + |P_{A\sim G}(y)| \le \epsilon$$

so $y \in \mathbf{B}_x(\epsilon)$ and (4) holds. □

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Proof. Let

$$\mathbf{B}_A = \{x \in \mathbf{H}_A : |x| \le 1\}.$$

Then $\mathbf{B}_A$ is closed and bounded. It is *not* totally bounded if A is infinite. That is because

$$|\mathbf{e}_\alpha - \mathbf{e}_\beta| = \sqrt{2} \text{ whenever } \alpha, \beta \in A \text{ and } \alpha \ne \beta$$

so that if $0 < r < \sqrt{2}$ then no two of the $\mathbf{e}_\alpha$'s can belong to any closed ball of radius r and so $\mathbf{H}_A$ is not contained in the union of a *finite* number of closed balls of radius r.

Discarding members of $\mathcal{C}$ whose diameter exceeds a given positive real number if necessary, we see we may assume without loss of generality that there is a positive real number R such that $\mathbf{diam}\, C \le R$ whenever $C \in \mathcal{C}$. Whenever F is a finite subset of A and $y \in C \in \mathcal{C}$ we may use the Cauchy-Schwartz inequality in $\mathbb{R}^F$ to infer that

$$\sqrt{\sum_{\alpha\in F} x_\alpha^2} \le \sqrt{\sum_{\alpha\in F} (x_\alpha - y_\alpha)^2} + \sqrt{\sum_{\alpha\in F} y_\alpha^2} \le \sqrt{|F|}\mathbf{diam}\, C + R.$$

Since we the diameter of C here can be made arbitrarily small, we find that $x \in \mathbf{H}_A$ and $|x| \le R$.

Suppose $\epsilon > 0$. We will show that

$$\text{(1)} \qquad \mathbf{B}_x(\epsilon) \cap C \ne \emptyset \text{ for any } C \in \mathcal{C}$$

and that will complete the proof. Choose $C_0 \in \mathcal{C}$ such that $\mathbf{diam}\, C \le \epsilon/4$. Choose $y \in C_0$. Choose a finite subset F of A such that

$$|P_{X\sim F}(x)| \le \epsilon/4 \text{ and } |P_{X\sim F}(y)| \le \epsilon/4.$$

Whenever $z \in C \in \mathcal{C}$ and $C \subset C_0$ we have

$$x - z = P_F(x-z) + P_{A\sim F}(x) + P_{A\sim F}(y-z) - P_{A\sim F}(y).$$

Using the Triangle Inequality we infer that

$$\begin{aligned}|x-z| &\le |P_F(x-z)| + |P_{A\sim F}(x)| + |P_{A\sim F}(y-z)| + |P_{A\sim F}(y)| \\ &\le |F|\mathbf{diam}\, C + \epsilon/4 + \mathbf{diam}\, C_1 + \epsilon/4 \\ &\le |F|\mathbf{diam}\, C + 3\epsilon/4 \\ &\le \epsilon\end{aligned}$$

provided $|F|\mathbf{diam}\, C \le \epsilon/4$. Thus (1) holds since $\mathcal{C}$ is nested. □