The Gauss-Bonnet Theorem for Surfaces.

1. Frame fields. Let n be a positive integer and let T be an open subset of some Euclidean space. Suppose the column vector

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 \\ \cdots \\ \mathbf{u}_n \end{bmatrix}$$

is a **smooth n-frame field on T** by which we mean that each $\mathbf{u}_i,\ i = 1, \ldots, n$ is a smooth $\mathbf{R}^n$-valued function on T and that $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_n$ never vanishes. Let

$$\theta$$

be the $n \times n$-matrix of smooth one forms on T determined by the requirement that

$$(1) \qquad d\mathbf{U} = \theta \mathbf{U};$$

θ is called the **connection matrix of U**. Applying exterior differentiation to (1) we find that

$$(2) \qquad d\theta_{i,j} = -\sum_{k=1}^{n} \theta_{i,k} \wedge \theta_{k,j}, \ \ k = 1, \ldots, n;$$

these equations are called the **structure equations for θ.**
Theorem. $\mathbf{U}$ is orthonormal if and only if θ is skewsymmetric.[1]
Proof. Straighforward exercise for the reader. □

Suppose $\mathbf{V}$ is another n-frame field on T and let η be its connection matrix. Let g be the smooth function with values in $\mathbf{GL}(\mathbf{R}^n)$ determined by the requirement that

$$(3) \qquad \mathbf{V} = g\mathbf{U}.$$

Applying d to (3) we obtain

$$\eta g \mathbf{U} = \eta \mathbf{V} = d\mathbf{V} = (dg)\mathbf{U} +_g d\mathbf{U} = (dg)\mathbf{U} + g\theta\mathbf{U};$$

multiplying on the right by g^{-1} we obtain

$$(4) \qquad \eta = (dg)g^{-1} + g\theta g^{-1}.$$

Of particular interest to us will be the following
Theorem. Suppose $n = 3$, T is simply connected and $\mathbf{u}_3 = \pm\mathbf{v}_3$. Then there is a smooth function $\alpha : T \to \mathbf{R}$ such that

$$(5) \qquad \eta_{1,2} = d\alpha + \theta_{1,2}.$$

Proof. Because T is simply connected there exists a smooth function $\alpha : T \to \mathbf{R}$ such that either

$$g = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

[1] This is the **first fundamental principle of differential geometry**. The **second fundamental principle of differential geometry** is that $dd = 0$ which amounts to the equality of mixed partial derivatives.

or this equation holds with its second column multiplied by -1. Since $\mathbf{O}(\mathbf{R}^2)$ is Abelian, we have $g\theta g^{-1} = \theta$; moreover, a simple calculation yields

$$(dg)g^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} d\alpha$$

and (5) now follows from (4). □

2. Surfaces in $\mathbf{R}^3$. Suppose $S \in \mathbf{M}_{2,3}$ and $X : T \to S$ is a local parameter for S. Suppose $\mathbf{U}$ is an orthonormal 3-frame field which is **adapted to** S by which we mean that

(1) $$dX \bullet \mathbf{u}_3 = 0.$$

Keeping in mind (1), we may define the smooth independent one forms $\omega_i,\ i = 1, 2$ on T by requiring that

(2) $$dX = \omega_1 \mathbf{u}_1 + \omega_2 \mathbf{u}_2.$$

Applying d to (2) we obtain the **first structure equations**

(3) $$d\omega_1 = \theta_{1,2} \wedge \omega_2, \quad d\omega_2 = \theta_{2,1} \wedge \omega_1$$

and the equation

(4) $$\theta_{3,1} \wedge \omega_1 + \theta_{3,2} \wedge \omega_2 = 0.$$

Letting the 2×2-matrix

$$b$$

of smooth functions on T be defined by requiring that

$$\begin{bmatrix} \theta_{3,1} \\ \theta_{3,2} \end{bmatrix} = b \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

we see that (4) is equivalent to

(5) $$b^t = b.$$

This amounts to saying that the derivative of a smooth unit normal field to S induces a symmetric linear transformation of the tangent space at each point of S. Let the smooth function

$$K : T \to \mathbf{R}$$

be defined by requiring that

$$d\theta_{1,2} = K\omega_1 \wedge \omega_2.$$

we call K the **Gauss curvature of X.** keeping in mind 1(5) we find that if Y is another local parameter for S and L is its Gauss curvature then $K \circ X^{-1} = L \circ Y^{-1}$ on the intersection of the ranges of X and Y. The structure equations 1(2) for θ amount to the **Gauss Curvature Equation** or **Theorem Egregium**

(6) $$K = \mathbf{det}\, b$$

and the **Codazzi-Mainardi Equations**

(7) $$d\theta_{3,1} = -\theta_{3,2} \wedge \theta_{2,1}, \quad d\theta_{3,2} = -\theta_{3,1} \wedge \theta_{1,2}.$$

As an illustration of these ideas, suppose b equals some smooth function λ times the identity which amounts to $\theta_{3,i} = \lambda\omega_i,\ i = 1,2$. Applying the first Codazzi-Mainardi equation and the first equation of (3) we find that

$$d\lambda \wedge \omega_1 + \lambda d\omega_1 \;=\; d(\lambda\omega_1) \;=\; d\theta_{3,1} \;=\; -\theta_{3,2} \wedge \theta_{2,1} \;=\; -\lambda\omega_2 \wedge \theta_{2,1} \;=\; \lambda d\omega_1$$

which implies that $d\lambda \wedge \omega_1 = 0$. Similarly, we find that $d\lambda \wedge \omega_2 = 0$. But this forces $d\lambda = 0$. Let us now assume that T is connected. It follows that λ is constant. If $\lambda \equiv 0$ then $d\mathbf{u}_3 = 0$ is constant so $d(X \bullet \mathbf{u}_3) = dX \bullet \mathbf{u}_3 = 0$ so the range of X lies in a plane. If $\lambda \neq 0$ then $d(\frac{\mathbf{u}_3}{R} - X) = \mathbf{0}$ so $\frac{\mathbf{u}_3}{R} - X = \mathbf{c}$ for some constant vector $\mathbf{c}$ so $|X - \mathbf{c}| = \frac{1}{|\lambda|}$ and X is spherical.

3. Geodesic curvature.

Let us retain the assumptions and notations of 2. Suppose $C \in \mathbf{M}^{\partial}_{1,3}$, C is oriented and is connected, C has finite length L and $\mathbf{cl}\, C \subset \mathbf{rng}\, X$. Then there is one and only on positively oriented local parameter $c : (0, L) \to C$ whose range equals C and there is one and only one orthonormal 3 frame field $\mathbf{V}$ on $(0, L)$ such that $\mathbf{v}_1 = c'$, $\mathbf{v}_1 \times \mathbf{v}_2 = (\mathbf{u}_1 \circ \mathbf{u}_2) \circ X^{-1} \circ c$ and $\mathbf{v}_3 = \mathbf{u}_3 \circ X^{-1} \circ c$. Let η be the connection form of $\mathbf{V}$ and let

$$\kappa : (0, L) \to \mathbf{R},$$

the **geodesic curvature of** C **in** S, be determined by the requirement that $\eta_{1,2} = \kappa\, ds$ where s is the identity function of $(0, L)$. We say C is **geodesic in** S if $\kappa = 0$ which amounts c'' being normal to S. Let

$$\alpha : (0, L) \to \mathbf{R}$$

be the smooth function such that

$$\mathbf{v}_1 \;=\; \cos\alpha\, (\mathbf{u}_1 \circ X^{-1} \circ c) + \sin\alpha\, (\mathbf{u}_1 \circ X^{-1} \circ c).$$

Note that any two such functions differ by a constant and that α can be extended to a smooth function whose domain contains the closure of $(0, L)$; in particular

$$\alpha_- \;=\; \lim_{s \downarrow 0} \alpha(s) \quad \text{and} \quad \alpha_+ \;=\; \lim_{s \uparrow L} \alpha(s)$$

exist. From our earlier work we find that

$$\eta_{1,2} \;=\; \theta_{1,2} \circ X^{-1} \circ c + d\alpha.$$

It follows that

$$\int_{(0,L)} \kappa\, ds \;=\; \int_{(0,L)} (X^{-1} \circ c)^{\#}\theta_{1,2} + \alpha_+ - \alpha_-.$$

Now suppose R is an open subset of $\mathbf{rng}\, S$ whose closure is a compact subset of $\mathbf{rng}\, X$ and whose boundary relative to $\mathbf{rng}\, X$ is the union of a finite set B and the union of a finite disjointed subfamily $\mathcal{C}$ of $\mathbf{M}^{\partial}_{1,3}$ each of whose members is connected and has finite length. Suppose, in addition, that each member of B is a member of exactly two of the sets $\{\partial C : C \in C\}$. For each $\mathbf{b} \in B$ let

$$\gamma_{\mathbf{b}} \;=\; \mathbf{length}\left(\mathbf{S}^2 \cap \mathbf{Tan}(R, \mathbf{b})\right).$$

Then we have the **Gauss-Bonnet formula**

$$\int K\, d||R|| \;=\; \sum_{C \in \mathcal{C}} \int_C \kappa_C\, d||C|| \;+\; \sum_{b \in B} \gamma_{\mathbf{b}}.$$

This is follows from the preceding and the fact that

$$\int_{X^{-1}[R]} d\theta_{1,2} \;=\; \int_{\partial X^{-1}[R]} \theta_{1,2}$$

which is Stokes' Theorem for a plane region with well behaved corners.