Differential Forms.

Suppose U is an open subset of $\mathbf{R}^n$.

Defintion. For each nonnegative integer p we let

$$\mathcal{A}^p(U)$$

be the set of smooth maps

$$\varphi : U \to \bigwedge^p \mathbf{R}^n.$$

Note that $\mathcal{A}^p(U)$ is a vector space in a natural way. Whenever f is a smooth real valued function on U and $\varphi \in \mathcal{A}^p(U)$ we define $f\varphi$ in $\mathcal{A}^p(U)$ by setting $f\varphi(x) = f(x)\varphi(x)$ for x in U.

Proposition. There is one and only one bilinear map from

$$\mathbf{W} : \mathbf{L}(\bigwedge^1 \mathbf{R}^n, \bigwedge^p \mathbf{R}^n) \to \bigwedge^{p+1} \mathbf{R}^n$$

such that

$$\mathbf{W}(\omega\varphi) = \omega \wedge \varphi \quad \text{whenever } (\omega, \varphi) \in \bigwedge^1 \mathbf{R}^n \times \bigwedge^p \mathbf{R}^n.$$

Moreover, if $\Phi \in \mathbf{L}(\bigwedge^1 \mathbf{R}^n, \bigwedge^p \mathbf{R}^n)$ then

$$\mathbf{W}(\Phi) = \sum_{i=1}^n \mathbf{v}^i \wedge \Phi(\mathbf{v}_i)$$

whenever $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an ordered basis for $\mathbf{R}^n$ and $\mathbf{v}^1, \ldots, \mathbf{v}^n$ is the corresponding ordered basis for $\bigwedge^1 \mathbf{R}^n$.
Proof. Straightforward exercise for the reader. $\Box$

Definition. Exterior Differentiation. We define

$$d : \mathcal{A}^p(U) \to \mathcal{A}^{p+1}(U)$$

on φ in $\mathcal{A}^p(U)$ by letting

$$d\varphi(x) = \mathbf{W}(\partial\varphi(\mathbf{x})) = \sum_{i=1}^n \mathbf{e}^i \wedge \partial_i \varphi(x) \quad \text{for } x \text{ in } U.$$

One frequently writes

$$d\varphi \text{ instead of } d(\varphi).$$

Theorem. We have

(1) $d(c\,\varphi) = c\,d\varphi$ whenever $c \in \mathbf{R}$ and $\varphi \in \mathcal{A}^p(U)$;

(2) $d(\varphi + \psi) = d\varphi + d\psi$ whenever $\varphi, \psi \in \mathcal{A}^p(U)$;

(3) $df = \partial f$ whenever f is a smooth function on U;

(4) $d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^p \varphi \wedge d\psi$ whenever $\varphi \in \mathcal{A}^p(U)$ and $\psi \in \mathcal{A}^q(U)$;

(5) $d(d\varphi) = \mathbf{0}$ whenever $\varphi \in \mathcal{A}^p(U)$.

Proof. The first three statements are trivial.

Let $\varphi,\ \psi$ be as in (4). Then

$$\begin{aligned} d(\varphi\wedge\psi) &= \sum_{i=1}^n \mathbf{e}^i \wedge \partial_i(\varphi\wedge\psi) \\ &= \sum_{i=1}^n \mathbf{e}^i \wedge (\partial_i\varphi\wedge\psi + \varphi\wedge\partial_i\psi) \\ &= (\sum_{i=1}^n \mathbf{e}^i \wedge \partial_i\varphi)\wedge\psi) + (-1)^p\varphi\wedge(\sum_{i=1}^n \mathbf{e}^i\wedge\partial_i\wedge\psi) \\ &= d\varphi\wedge\psi + (-1)^p\varphi\wedge d\psi. \end{aligned}$$

Let φ be as in (5). Then

$$\begin{aligned} d(d\varphi) &= \sum_{i=1}^n \mathbf{e}^i\wedge\partial_i(\sum_{j=1}^n \mathbf{e}^j\wedge\partial_j\varphi) \\ &= \sum_{i,j=1}^n \mathbf{e}^i\wedge\mathbf{e}^j\wedge\partial_i\partial_j\varphi \\ &= \mathbf{0} \end{aligned}$$

because $\partial_i\partial_j\varphi = \partial_j\partial_i\varphi$ and $\mathbf{e}^i\wedge\mathbf{e}^j = -\mathbf{e}^j\wedge\mathbf{e}^i$ whenever $i,j=1,\dots,n$. □

Pullbacks.

Suppose V is an open subset of some Euclidean space and $F : U \to V$ is smooth. For each nonnegative integer p we define

$$F^\# : \mathcal{A}^p(V) \to \mathcal{A}^p(U)$$

at $\varphi \in \mathcal{A}^p(V)$ by setting

$$F^\#\varphi(x) = \bigwedge\nolimits^p \partial F(x)(\varphi(F(x))) \quad \text{whenever } x \in U.$$

One readily verifies that

(1) $F^\#\varphi = \varphi\circ F$ if $p = 0$;

(2) $F^\#(c\,\varphi) = c\,F^\#\varphi$ whenever $c \in \mathbf{R}$ and $\varphi \in \mathcal{A}^p(V)$;

(3) $F^\#(\varphi+\psi) = F^\#\varphi + F^\#\psi$ whenever $\varphi,\psi \in \mathcal{A}^p(V)$;

(4) $F^\#(\varphi\wedge\psi) = F^\#\varphi\wedge F^\#\psi$ whenever $\varphi\in\mathcal{A}^p(V)$ and $\psi\in\mathcal{A}^q(V)$;

Suppose W is an open subset of some Euclidean space and $G : V \to W$ is smooth. Using the Chain Rule one finds that

(5) $(G\circ F)^\#\varphi = F^\#(G^\#\varphi)$ whenever $\varphi \in \mathcal{A}^p(W)$.

Theorem. Suppose V is an open subset of some Euclidean space and $F : U \to V$ is smooth. For each nonnegative integer p we have

$$d(F^\#\varphi) = F^\#(d\varphi) \quad \text{whenever } \varphi \in \mathcal{A}^p(V).$$

Proof. Note first that if f is a smooth real valued function on V then

$$d(F^\# f) = d(f\circ F) = \partial(f\circ F) = F^\#(df)$$

by the Chain Rule.

Suppose $\varphi = g\, df_1 \wedge \cdots \wedge df_p$ where $g, f_1, \ldots, f_p$ are smooth real valued functions on V. Then

$$d(F^{\#}(df_i)) = d(d(F^{\#} f_i)) = 0 \quad \text{for } i = 1, \ldots, p$$

so

$$\begin{aligned} d(F^{\#}\varphi) &= d(F^{\#} g\, F^{\#}(df_1) \wedge \cdots \wedge F^{\#}(df_p)) \\ &= d(F^{\#} g) \wedge F^{\#}(df_1) \wedge \cdots \wedge F^{\#}(df_p) \\ &= F^{\#}(dg) \wedge F^{\#}(df_1) \wedge \cdots \wedge F^{\#}(df_p) \\ &= F^{\#}(dg \wedge df_1 \wedge \cdots \wedge df_p) \\ &= F^{\#}(\varphi). \end{aligned}$$

Since any member of $\mathcal{A}^p(V)$ is a sum of forms of this type, the Theorem follows from the additivity of d and $F^{\#}$. $\square$