Vector fields and divergence.

Let U be an open subset of $\mathbf{R}^n$.

Definition. We let

$$\mathcal{F}(U)$$

be the algebra of smooth real valued functions on U and we let

$$\mathcal{X}(U)$$

be the $\mathcal{F}(U)$ module of smooth $\mathbf{R}^n$ valued functions on U. We call the members of $\mathcal{X}(U)$ **(smooth) vector fields on** U. We let

$$\mathcal{F}_c(U) \quad \text{and} \quad \mathcal{X}_c(U)$$

be the members of $\mathcal{F}(U)$ and $\mathcal{X}(U)$, respectively, whose support is a compact subset of U.

Theorem. Suppose $X \in \mathcal{X}(U)$ and ϕ is its flow. Then

$$\frac{d}{dt} \mathbf{det}\ \partial\phi_t(\mathbf{x})\Big|_{t=0} = \mathbf{trace}\,\partial X(\mathbf{x}) \quad \text{for } \mathbf{x} \in U.$$

Proof. Exercise for the reader. Use the Chain Rule and the fact that

$$\partial\,\mathbf{det}\,(i_{\mathbf{R}^n}) = \mathbf{trace}$$

which we have already proved. To prove this last formula, observe that for any $H \in \mathbf{L}(\mathbf{R}^n; \mathbf{R}^n)$ we have

$$\frac{d}{dt}\mathbf{det}\,\mathbf{i}_{\mathbf{R}^n} + tH\Big|_{t=0} \frac{d}{dt}(\mathbf{e}_1 + tH(\mathbf{e}_1)) \wedge \cdots \wedge (\mathbf{e}_n + tH(\mathbf{e}_n))\Big|_{t=0}$$

which you can evaluate using Leibniz' Rule. $\square$

Definition. For each $X \in \mathcal{X}(U)$ we set

$$\mathbf{div}\,X = \mathbf{trace}\,\partial X \in \mathcal{F}(U).$$

Theorem. Suppose $X \in \mathcal{X}(U)$, $\rho \in \mathcal{F}(U)$ and K is a compact subset of U. Then

$$\frac{d}{dt}\int_{\phi_t[K]} \rho\,\Big|_{t=0} = \int_K \mathbf{div}\,(\rho\,X).$$

Proof. Exercise for the reader. Use the Change of Variables Formula for Multiple Integrals and the previous Theorem. $\square$

Definition. We define the $\mathcal{F}(U)$-isomorphisms

$$\beta : \mathcal{X}(U) \to \mathcal{A}^1(U) \quad \text{and} \quad \gamma : \mathcal{X}(U) \to \mathcal{A}^{n-1}(U)$$

at $X \in \mathcal{X}(U)$ by setting

$$\beta(X)(\mathbf{x})(\mathbf{u}) = X(\mathbf{x}) \bullet \mathbf{u} \quad \text{whenever } \mathbf{x} \in U \text{ and } \mathbf{u} \in \mathbf{R}^n$$

and by setting

$$\gamma(X) = \iota_X(\mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n).$$

Proposition. Suppose $f \in \mathcal{F}(U)$. Then

$$df = \beta(\mathbf{grad}\,f).$$

Proof. This is transparent. $\square$

Proposition. Suppose $X \in \mathcal{X}(U)$. Then

$$d\gamma(X) \;=\; (\mathbf{div}\, X)\, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n.$$

Proof. Exercise for the reader. $\square$

The curl in $\mathbf{R}^2$.

Suppose $n = 2$.

Definition. We define

$$\mathbf{curl} : \mathcal{X}(U) \to \mathcal{F}(U)$$

at $X \in \mathcal{X}(U)$ by requiring that

$$\mathbf{curl}\, X\, \mathbf{e}^1 \wedge \mathbf{e}^2 \;=\; d(\beta(X)).$$

Exercise. Show that

$$\mathbf{curl}\, X \;=\; -\mathbf{div}\, X \quad \text{whenever } X \in \mathcal{X}(U).$$

The curl in $\mathbf{R}^3$.

Suppose $n = 3$.

Definition. We define

$$\mathbf{curl} : \mathcal{X}(U) \to \mathcal{X}(U)$$

at $X \in \mathcal{X}(U)$ by requiring that

$$\iota_{\mathbf{curl}\, X}(\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3) \;=\; d\beta(X) \quad \text{whenever } X \in \mathcal{X}(U).$$

Exercise. Write a formula for the curl in terms of the standard components of $X \in \mathcal{X}(U)$.
Remark. It follows from $dd = 0$ that $\mathbf{curl}\,\mathbf{grad} = 0$ and that $\mathbf{div}\,\mathbf{curl} = 0$.