1. Differentiation of vector valued functions of a real variable.

Definition 1.1. Suppose $A \subset \mathbb{R}$, E is a normed vector space,

$$f : A \to E.$$

We let

$$f' = \left\{ (a, m) : a \in \textbf{int}\, A \text{ and } m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right\}.$$

Note that f' is a function. We say f is **differentiable at** a if a is in the domain of f'. For each nonegative integer m we define $f^{(m)}$ by setting $f^{(0)} = f$, $f^{(1)} = f'$ and requiring that $f^{(m+1)} = (f^{(m)})'$.

Theorem 1.1. Suppose $A \subset \mathbb{R}$, E is a normed vector space, $f : A \to E$ and f is differentiable at a. Then

$$\lim_{x \to a} f(x) = f(a).$$

That is, f is continuous at a.

Proof. We give two proofs. The first use rules for limits and the second uses ϵ and δ.

1st Proof. We have

$$f(x) = (\frac{f(x) - f(a)}{x - a})(x - a) + f(a) \quad \text{for } a \in A.$$

Moreover,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a), \quad \lim_{x \to a} x - a = 0 \quad \text{and} \quad \lim_{x \to a} f(a) = f(a).$$

It follows from the rules for limits that

$$\lim_{x \to a} f(x) = f'(a)0 + f(a) = f(a).$$

2nd Proof. Let $\epsilon > 0$. There is $\eta > 0$ such that

$$x \in A \text{ and } 0 < |x - a| < \eta \;\Rightarrow \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| \leq 1.$$

Let

$$\delta = \min \left\{ \frac{\epsilon}{1 + |f'(a)|}, \eta \right\}.$$

If $x \in A$ and $0 < |x - a| < \delta$ then

$$\begin{aligned} |f(x) - f(a)| &= \left(\frac{f(x) - f(a)}{x - a} - f'(a) \right)(x - a) + f'(a)(x - a) \\ &\leq \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| |x - a| + |f'(a)||x - a| \\ &\leq (1 + |f'(a)|)|x - a| \\ &< \epsilon. \end{aligned}$$

$\square$

Theorem 1.2. Suppose $A \subset \mathbb{R}$,

$$f : A \to \mathbb{R},$$

f is differentiable at a and *either*

$$f(x) \le f(a) \quad \text{whenever } x \in A$$

or

$$f(x) \ge f(a) \quad \text{whenever } x \in A.$$

Then

$$f'(a) = 0.$$

Remark 1.1. Note that

$$\frac{g \circ f(x) - g \circ f(a)}{x - a} = \frac{g \circ f(x) - g \circ f(a)}{f(x) - f(a)} \frac{f(x) - f(a)}{x - a}$$

whenever $x \in A \sim \{a\}$ and $f(x) \in B \sim \{b\}$.

Proof. Let $\epsilon > 0$. Choose $\delta > 0$ such that

$$(1) \qquad x \in A \text{ and } 0 < |x - a| < \delta \;\Rightarrow\; \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \epsilon.$$

This amounts to

$$(1) \quad x \in A \text{ and } 0 < |x - a| < \delta \;\Rightarrow\; \frac{f(x) - f(a)}{x - a} - \epsilon < f'(a) < \frac{f(x) - f(a)}{x - a} + \epsilon.$$

Suppose $f(x) \le f(a)$ whenever $x \in A$. Then (1) amounts to

$$x \in A \text{ and } 0 < |x - a| < \delta \;\Rightarrow\; \frac{f(x) - f(a)}{x - a} - \epsilon < f'(a) < \frac{f(x) - f(a)}{x - a} + \epsilon.$$

Keeping in mind that $a \in \textbf{acc}\, A$ we choose $x \in A \cap (a, a - \delta)$ to infer that $-\epsilon < f'(a)$ and choose $x \in A \cap (a, a + \delta)$ to infer that $f'(a) < \epsilon$. Since ϵ is arbitrary we conclude that $f'(a) = 0$.

Suppose $f(x) \ge f(a)$ whenever $x \in A$. Then (1) amounts to

$$x \in A \text{ and } 0 < |x - a| < \delta \;\Rightarrow\; f'(a) - \epsilon < \frac{f(x) - f(a)}{x - a} < f'(a) + \epsilon.$$

Keeping in mind that $a \in \textbf{acc}\, A$ we choose $x \in A \cap (a, a - \delta)$ to infer that $-\epsilon < f'(a)$ and choose $x \in A \cap (a, a + \delta)$ to infer that $f'(a) < \epsilon$. Since ϵ is arbitrary we conclude that $f'(a) = 0$.

Alternatively, having dealt with one of these cases we can replace f by $-f$ to handle the other case. □

Theorem 1.3. (The Chain Rule) Suppose

(i) $a \in A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and f is differentiable at a.
(ii) $b \in B \subset \mathbb{R}$, $g : B \to \mathbb{R}$ and g is differentiable at b
(iii) $b = f(a)$.

Then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) \;=\; g'(f(a)) f'(a).$$

Proof. **1st Proof.** Since $b \in \textbf{int}\, B$ there is an open subset V of $\mathbb{R}$ such that $b \in V \subset B$. Since f is continuous at a there is an open subset U of $\mathbb{R}$ such that $a \in U$ and $f[A \cap U] \subset V$. Since $a \in \textbf{int}\, A$ it follows that $a \in \textbf{int}\,\textbf{dmn}\, g \circ f$.

Let $s : B \to \mathbb{R}$ be such that

$$s(y) = \begin{cases} \frac{g(y)-g(b)}{y-b} - g'(b) & \text{if } y \in B \text{ and } y \neq b \\ 0 & \text{if } y = b. \end{cases}$$

Then s is continuous at b. If $x \in \textbf{dmn}\, g \circ f$ and $x \neq a$ we have

$$\frac{g(f(x)) - g(f(a))}{x-a} = g'(f(a))\frac{f(x)-f(a)}{x-a} + s(f(x))\frac{f(x)-f(a)}{x-a}. \tag{4}$$

By previous theory,

$$\lim_{x \to a} s(f(x)) = s(\lim_{x\to a} f(x)) = s(f(a) = 0.$$

We complete the proof by letting $x \to a$ in (4) and using limit rules. proof.

2nd Proof. Set

$$O(x) = g \circ f(x) - g \circ f(a) - g'(f(a))f'(a)(x-a) \quad \text{for } x \in A.$$

Having already shown that a is an interior point of the domain of $g \circ f$ the statement to be proved is equivalent to

$$\lim_{x \to a} \frac{O(x)}{x-a} = 0.$$

Set

$$N(y) = g(y) - g(f(a)) - g'(f(a))(y - f(a)) \quad \text{for } y \in B$$

and set

$$M(x) = f(x) - f(a) - f'(a)(x-a) \quad \text{for } x \in A.$$

For any $x \in A$ we have

$$\begin{aligned} O(x) &= g \circ f(x) - g \circ f(a) - g'(f(a))f'(a)(x-a) \\ &= g(f(x)) - g(f(a)) - g'(f(a))(f(x) - f(a)) \\ &\qquad + g'(f(a))\,[f(x) - f(a) - f'(a)(x-a)] \\ &= N(f(x)) + g'(f(a))M(x). \end{aligned}$$

Suppose $0 < \eta < \epsilon$. Since g is differentiable at $f(a)$ there is δ_g such that

$$y \in B \text{ and } 0 < |y - g(f(a))| < \delta_g \;\Rightarrow\; |N(y)| \leq \frac{\eta}{2(1+|f'(a)|)}|y - f(a)|.$$

Since f is differentiable at a there is δ_f such that

$$a \in A \text{ and } 0 < |x-a| < \delta_f \;\Rightarrow\; \text{ and } |M(x)| \leq \min\left\{\frac{\eta}{2(1+|g'(f(a))|}, 1\right\}|x-a|.$$

Let

$$\delta = \min\left\{\delta_f, \frac{\delta_g}{1+|f'(a)|}\right\}.$$

Suppose $x \in A$ and $0 < |x-a| < \delta$. Then

$$|f(x) - f(a)| \leq |M(x)| + |f'(a)||x-a| \leq (1+|f'(a)|)|x-a| < \delta_g.$$

But then

$$|N(f(x))| \leq \frac{\eta}{2(1+|f'(a)|)}|f(x) - f(a)| \leq \frac{\eta}{2}|x-a|$$

and

$$|g'(f(a))||M(x)| \le |g'(f(a))|\frac{\eta}{2(1+|g'(f(a)))|}|x-a| \le \frac{\eta}{2}|x-a|.$$

Consequently,

$$|O(x)| \le |N(f(x))| + |g'(f(a))||M(x)| \le \frac{\eta}{2} + \frac{\eta}{2} < \epsilon,$$

as desired. □

Theorem 1.4. (The Intermediate Value Theorem.) Suppose I is an interval in $\mathbb{R}$ and

$$f : I \to \mathbb{R}$$

is continuous. Then $\mathbf{rng}\, f$ is an interval.

Proof. This follows immediately from the fact that a subset of $\mathbb{R}$ is connected if and only if it is an interval and that fact that the continuous image of a connected set is connected. □

Corollary 1.1. Suppose I is an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ is continuous. Then f is univalent if and only if either f is increasing or f is decreasing. Moreover, if f is univalent then f^{-1} is continuous.

Proof. Exercise for the reader. □

Theorem 1.5. Suppose

(i) I is an open interval in $\mathbb{R}$, $a \in I$ and $f : I \to \mathbb{R}$ is continuous and univalent;
(ii) $b = f(a)$, $B \subset \mathbb{R}$ and $g : B \to \mathbb{R}$;
(iii) $M \in \mathbb{R}$ and $\lim_{x\to a} g(f(x)) = M$.

Then $\lim_{y\to b} g(y)$ exists and equals M.

Proof. Since f is continuous at a, since a is an accumulation point of the domain of $g \circ f$ by (3) and since f is univalent we infer that b is an accumulation point of B.

Let $\epsilon > 0$. Choose $\delta > 0$ such that $(a-\delta, a+\delta) \subset I$ and

$$x \in \mathbf{dmn}\, g \circ f \text{ and } |x-a| < \delta \;\Rightarrow\; |g(f(x) - g(f(a))| < \epsilon.$$

By virtue of our previous theory, $f[(a-\delta, a+\delta)]$ is an open interval so there is $\eta > 0$ such that $(b-\eta, b+\eta) \subset f[(a-\delta, a+\delta)]$. Suppose $|y-b| < \eta$. There is a unique $x \in (a-\delta, a+\delta)$ such that $y = f(x)$. So if $y \in B$ then $|g(y) - M| = |g(f(x)) - M| < \epsilon$. □

Theorem 1.6. Suppose I is an interval in $\mathbb{R}$,

$$f : I \to \mathbb{R}$$

is continuous and univalent, $a \in \mathbf{int}\, I$,f is differentiable at a and $f'(a) \neq 0$. Then f^{-1} is differentiable at $f(a)$ and

$$(f^{-1})'(f(a)) = 1/f'(a).$$

Proof. Let $J = \{f(x) : x \in \mathbf{int}\, I\}$. By virtue of the preceding theory, J is an open interval. Let

$$g(y) \;=\; \frac{f^{-1}(y) - a}{y - f(a)} \quad \text{for } y \in J \sim \{f(a)\}.$$

For $x \in I \sim \{a\}$ we have that

$$g(f(x)) \;=\; \frac{x-a}{f(x)-f(a)}$$

has limit $1/f'(a)$ as $x \to a$. We may now complete the proof by making use of the previous Theorem. □

Theorem 1.7. (The Mean Value Theorem.) Suppose $a, b \in \mathbb{R}$, $a < b$,

$$f : [a,b] \to \mathbb{R},$$

f is continuous and f is differentiable at each point of (a,b). Then there is a point $\xi \in (a,b)$ such that

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}.$$

Proof. Define $g : [a,b] \to \mathbb{R}$ by letting

$$g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a) \quad \text{for } x \in [a,b].$$

Note that g is continuous, that $g(a) = 0 = g(b)$ and that g is differentiable at each point $x \in (a,b)$ with

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

Since $[a,b]$ is compact, there is some point $\xi_{max} \in [a,b]$ such that $g(x) \leq g(\xi_{max})$ whenever $x \in [a,b]$ and there is some point $\xi_{min} \in [a,b]$ such that $g(\xi_{max}) \leq g(x)$ whenever $x \in [a,b]$. If g is constant the Theorem holds trivially, so let us assume g is nonconstant. Then at least one of ξ_{max}, ξ_{min} is in (a,b) and, by the previous Theorem, is a point where g' vanishes. □

Corollary 1.2. Suppose $a, b \in \mathbb{R}$, $a < b$, E is a normed vector space,

$$f : [a,b] \to E,$$

f is continuous, f is differentiable at each point of (a,b), $0 \leq M < \infty$ and

$$|f'(t)| \leq M \quad \text{whenever } t \in (a,b).$$

Then

$$|f(b) - f(a)| \leq M(b-a).$$

Proof. Suppose ω is a bounded real valued linear function on E. Applying the Mean Value Theorem to $\omega \circ f$ we infer that

$$|\omega(f(b)-f(a))| \leq ||\omega|| M(b-a).$$

Our assertion now follows from the Hahn-Banach Theorem. □

Theorem 1.8. (Taylor's Theorem with Lagrange form for the remainder.) Suppose n is a positive integer, I is an open interval of real numbers,

$$f : I \to \mathbb{R}$$

is $n+1$ times differentiable at each point of I and $a \in I$. Let

$$P(x) \;=\; \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}(x-a)^m \qquad \text{for each } x \in I.$$

Then for each $x \in I \sim \{a\}$ there is a real number ξ strictly between a and x such that

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

In particular, if M is a nonnegative real number with the property that

$$|f^{(n+1)}(x)| \le M \qquad \text{for each } x \in I$$

then

$$|f(x) - P(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for each } x \in I.$$

Proof. Suppose $a < x \in I$; it will be obvious how to modify the proof to hande the case when $x \in I$ and $x < a$.

Lemma 1.1. Suppose $\phi : I \to \mathbb{R}$, ϕ is $n+1$ times differentiable on I, $\phi^{(m)}(a) = 0$ for $0 \le m \le n$ and $\phi(x) = 0$. Then there is $\xi \in (a,x)$ such that $\phi^{(n+1)}(\xi) = 0$.

Proof. Induct on n. The Lemma follows directly from the Mean Value Theorem in case $n = 0$. Suppose $n > 0$. By the Mean Value Theorem there is $\eta \in (a,x)$ such that $\phi'(\eta) = 0$. Now apply induction with ϕ and x replaced by ϕ' and η. $\square$

Let

$$R(t) = f(t) - P(t) \quad \text{for } t \in I$$

and let

$$\phi(t) = R(t) - R(x)\left(\frac{t-a}{x-a}\right)^{n+1} \qquad \text{for } t \in I.$$

Evidently, $\phi^{(m)}(a) = 0$ for $0 \le m \le n$ and $\phi(x) = 0$. By the Lemma there is $\xi \in (a,x)$ such that $\phi^{(n+1)}(\xi) = 0$. Since

$$\phi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - (n+1)!R(x)$$

the Theorem is proved. $\square$

Theorem 1.9. Suppose I is a nonempty open interval, $f : I \to \mathbb{R}$ and f is differentiable at each point of I. Then

$$\left\{\frac{f(y)-f(x)}{y-x} : x, y \in I \text{ and } x \ne y\right\}$$

is an interval. Moreover $\mathbf{rng}\, f'$ is an interval.

Proof. Let $U = \{(x,y) \in I \times I : x < y\}$. We define $g : U \to \mathbb{R}$ by setting

$$g(x,y) = \frac{f(y)-f(x)}{y-x} \quad \text{for } (x,y) \in U.$$

Since U is connected and g is continuous the range of g is connected and, therefore, an interval. It follows from the Mean Value Theorem that $\mathbf{rng}\, g \subset \mathbf{rng}\, f'$. Since $\mathbf{rng}\, f' \subset \mathbf{cl}\,\mathbf{rng}\, g$ we find that $\mathbf{rng}\, f'$ is connected. $\square$