The change of variables formula for multiple integrals.

Let n be a positive integer.

Theorem. Suppose U is an open subset of $\mathbf{R}^n$,

$$f : U \to \mathbf{R}^n$$

and the following conditions hold:

(i) f is continuously differentiable;

(ii) f is univalent and

(iii) $\mathbf{ker}\, \partial f(a) = \{\mathbf{0}\}$ whenever $a \in U$.

Then

$$\mathcal{L}^n(f[A]) = \int_A |\mathbf{det}\, \partial f(x)|\, dx$$

whenever A is a Lebesgue measurable subset of U.

Proof. We set

$$||x|| = \max\{|x_i| : i = 1, \ldots, n\} \quad \text{for each } x \in \mathbf{R}^n$$

and note that $||\cdot||$ is a norm on $\mathbf{R}^n$. We let $|||\cdot|||$ be the corresponding norm on $\mathbf{L}(\mathbf{R}^n; \mathbf{R}^n)$; that is, for each $l \in \mathbf{L}(\mathbf{R}^n, \mathbf{R}^n)$ we set

$$|||l||| = \sup\{||l(v)|| : v \in \mathbf{R}^n \text{ and } ||v|| = 1\}.$$

For each compact cube C such that $C \subset U$ we let $a(C)$ be the center of S; we let $R(C)$ be the halfsidelength of C; we note that

$$C = \{x \in \mathbf{R}^n : ||x - a|| \leq R(C)\};$$

we let

$$\alpha(C) = \sup\{|||\partial f(x) - \partial f(a(C))||| : x \in C\};$$

and we let

$$\beta(C) = \inf\{|||\partial f(a(C))(u)||| : u \in \mathbf{R}^n \text{ and } ||u|| = 1\}.$$

Owing to simple approximation arguments we may assume that A is a compact cube. For each $\delta > 0$ let

$$\mathbf{A}(\delta) = \sup\{|||\partial f(x) - \partial f(a)||| : x, a \in A \text{ and } |||x - a||| \leq \delta\}$$

. Since ∂f is continuous and A is compact we find that

$$\lim_{\delta \downarrow 0} \mathbf{A}(\delta) = 0. \tag{2}$$

For each $x \in U$ let $\mathbf{b}(x) = \inf\{|||\partial f(x)(u)||| : ||u|| = 1\}$. Since $\mathbf{b}(x) = |||\partial f(x)^{-1}|||^{-1}$ for each $x \in U$ we find that that $\mathbf{b}$ is a positive continuous function on U. We set

$$\mathbf{B} = \inf\{\mathbf{b}(x) : x \in A\}. \tag{1}$$

Since $\mathbf{b}$ is continuous and A is compact we infer that $\mathbf{B} > 0$ so we may choose $\delta_0 > 0$ such that

$$\mathbf{A}(\delta_0) < \mathbf{B}.(3)$$

Suppose $0 < \eta \leq \delta_0$. We let $\mathcal{C}$ be a family of compact cubes with the following properties:

(4) $A = \cup \mathcal{C}$;

(5) $||C \cap D||_n = 0$ whenever $C, D \in \mathcal{C}$;

(6) the halflength of any side of any member of $\mathcal{C}$ does not exceed η.

Suppose $C \in \mathcal{C}$. Because

$$\alpha(C) \leq \mathbf{A}(\delta_0) < \mathbf{B} \leq \beta(C)$$

we may apply the Inverse Function Theorem with f, L, a, R there equal $f|C, \partial f(a(C)), a(C), R(C)$, respectively, to conclude that

$$\begin{aligned}\{f(a(C)) + \partial f(a(C))(h) : (1 - \alpha(C)/\beta(C))||h|| \leq R(C)\}& \\ \subset f[C]& \\ \subset \{f(a(C)) + \partial f(a(C))(h) : (1 + \alpha(C)/\beta(C))||h|| \leq R(C)\}&.\end{aligned}$$

Combining this with our earlier results about how areas change under linear maps we infer that

$$\begin{aligned}(1 - \alpha(C)/\beta(C))^n |\mathbf{det}\, \partial f(a(C))|\, ||C||_n& \\ = \mathcal{L}^n(\{f(a(C)) + \partial f(a(C))(h) : (1 - \alpha(C)/\beta(C))||h|| \leq R(C)\})& \\ \leq \mathcal{L}^n(f[C])& \\ \leq \mathcal{L}^n(\{f(a(C)) + \partial f(a(C))(h) : (1 + \alpha(C)/\beta(C))||h|| \leq R(C)\})& \\ = (1 + \alpha(C)/\beta(C))^n |\mathbf{det}\, \partial f(a(C))|\, ||C||_n&.\end{aligned}$$

Setting

$$I = \sum_{C \in \mathcal{C}} |\mathbf{det}\, \partial f(a(C))|\, ||C||_n$$

and keeping in mind (2) and (3) we find that

$$\big(1 - \mathbf{A}(\eta)/\mathbf{B}\big)^n I \leq \sum_{C \in \mathcal{C}} \mathcal{L}^n(f[C]) \leq \big(1 + \mathbf{A}(\eta)/\mathbf{B}\big)^n I.$$

Because f is univalent we find that

$$\sum_{C \in \mathcal{C}} \mathcal{L}^n(f[C]) = \sum_{C \in \mathcal{C}} \mathcal{L}^n(f[\mathbf{int}\, C]) = \mathcal{L}^n\left(f[\cup_{C \in \mathcal{C}} \mathbf{int}\, C]\right) = \mathcal{L}^n(f[A]).$$

Since I is a Riemann sum for $J = \int_A |\mathbf{det}\, \partial f(x)|\, dx$ with respect to cubes of halfsidelength η and since (2) holds we infer that $J = {}^n(f[A])$, as desired. □