Contents

1. The Riemann and Lebesgue integrals.

Fix a positive integer n. Recall that

$$\mathcal{R}_n \quad \text{and} \quad \mathcal{M}_n$$

are the family of rectangles in $\mathbb{R}^n$ and the algebra of multirectangles in $\mathbb{R}^n$, respectively.

Definition 1.1. We let

$$\mathcal{F}_n^+, \quad \mathcal{F}_n, \quad \mathcal{B}_n,$$

be the set of $[0,\infty]$ valued functions on $\mathbb{R}^n$; the vector space of real valued functions on $\mathbb{R}^n$; the vector space of $f \in \mathcal{F}_n$ such that $\{f \neq 0\} \cup \mathbf{rng}\, f$ is bounded, respectively.

We let

$$\mathcal{S}_n = \mathcal{B}_n \cap \mathcal{S}(\mathcal{M}_n) \subset \mathcal{F}_n \quad \text{and we let} \quad \mathcal{S}_n^+ = \mathcal{S}_n \cap \mathcal{F}_n^+ \subset \mathcal{S}^+(\mathcal{M}_n).$$

Thus $s \in \mathcal{S}_n$ if and only if $s : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\mathbf{rng}\, s$ is finite, $\{s = y\}$ is a multirectangle for each $y \in \overline{\mathbb{R}}$ and $\{s \neq 0\}$ is bounded and $s \in \mathcal{S}_n^+$ if and only if $s \in \mathcal{S}_n$ and $s \geq 0$.

We let $\mathcal{S}_{n,\uparrow}^+$ be the set of nondecreasing sequences in $\mathcal{S}_n^+$. For each $s \in \mathcal{S}_{n,\uparrow}^+$ we let

$$\sup s \in \mathcal{F}_n^+$$

be such that

$$\sup s(x) = \sup\{s_\nu(x) : \nu \in \mathbb{N}\} \quad \text{for } x \in \mathbb{R}^n$$

and we let

$$I_{n,\uparrow}^n(s) = \sup\{I_n^+(s_\nu) : \nu \in \mathbb{N}^+\}.$$

Remark 1.1. We shall prove below the nontrivial Theorem that if $s, t \in \mathcal{S}_{n,\uparrow}^+$ and $\sup s = \sup t$ then $I_{n,\uparrow}^n(s) = I_{n,\uparrow}^n(t)$.

Proposition 1.1. Suppose $c \in [0,\infty)$ and $s, t \in \mathcal{S}_{n,\uparrow}^+$. Then

(i) $cs \in \mathcal{S}_{n,\uparrow}^+$ and $I_{n,\uparrow}^n(cs) = cI_{n,\uparrow}^n(s)$;
(ii) $s + t \in I_{n,\uparrow}^n$ and $I_{n,\uparrow}^n(s+t) = I_{n,\uparrow}^n(s) + I_{n,\uparrow}^n(t)$;
(iii) if $s \leq t$ then $I_{n,\uparrow}^n(s) \leq I_{n,\uparrow}^n(t)$.

Proof. Straightforward exercise for the reader. □

Definition 1.2. For each $f \in \mathcal{F}_n^+$ we let

$$\mathbf{r}(f) = \inf\{I_n^+(s) : s \in \mathcal{S}_n^+ \text{ and } f \le s\}$$

and we let

$$\mathbf{l}(f) = \inf\{I_{n,\uparrow}^n(s) : s \in \mathcal{S}_{n,\uparrow}^+ \text{ and } f \le \sup s\}.$$

Proposition 1.2. We have

$$\mathbf{r}(s) = I_n^+(s) \quad \text{whenever } s \in \mathcal{S}_n^+.$$

Proof. This should be obvious. □

Corollary 1.1. We have

$$|I_n(s)| \le \mathbf{r}(|s|) \quad \text{whenever } s \in \mathcal{S}_n.$$

Proof. Indeed, for any $s \in \mathcal{S}_n$ we have $|I_n(s)| \le I_n^+(|s|)$. □

Remark 1.2. We also have

$$\mathbf{l}(s) = I_n^+(s) \quad \text{whenever } s \in \mathcal{S}_n^+.$$

We shall prove this nontrivial fact shortly.

Proposition 1.3. Suppose $f \in \mathcal{F}_n^+$ and $\mathbf{r}(f) < \infty$. Then $f \in \mathcal{B}_n$.

Proof. There is $s \in \mathcal{S}_n^+$ such that $f \le s$ and this implies $\mathbf{rng}\, f \cup \{f > 0\} \subset \mathbf{rng}\, s \cup \{s > 0\}$. □

Remark 1.3. On the other hand, if $a \in \mathbb{R}^n$ and $f = \infty 1_{\{a\}} \in \mathcal{F}_n^+$ and $f = \sup_\nu \nu 1_{\{a\}}$ so $\mathbf{l}(f) = 0$.

Proposition 1.4. We have

$$\mathbf{l} \le \mathbf{r}.$$

Proof. Suppose $f \in \mathcal{F}_n^+$, $s \in \mathcal{S}_n^+$ and $f \le s$. Let t be the sequence in $\mathcal{S}_n^+$ whose range equals s; that is, $t_\nu = s$ for all $\nu \in \mathbb{N}$. Then $\sup t = s$ so

$$\mathbf{l}(f) \le I_{n,\uparrow}^n(t) = I_n^+(s)$$

which is to say $\mathbf{l}(f)$ is a lower bound for the set of $I_n^+(u)$ corresponding to $t \in \mathcal{S}_n^+$ with $f \le t$. □

Proposition 1.5. $\mathcal{F}_n \ni f \mapsto \mathbf{r}(|f|)$ and $\mathcal{F}_n \ni f \mapsto \mathbf{l}(|f|)$ are extended seminorms on $\mathcal{F}_n$.

Proof. Straightforward exercise for the reader. □

Example 1.1. Let

$$Q = (0,1) \cap \mathbb{Q}.$$

We will show that

$$\mathbf{r}(1_Q) = 1 \quad \text{and that} \quad \mathbf{l}(1_Q) = 0.$$

Since $1_Q \le 1_{(0,1)} \in \mathcal{S}_n^+$ we find that

$$\mathbf{r}(1_Q) \le I_n^+(1_{(0,1)}) = ||(0,1)|| = 1.$$

Suppose $1_Q \le s \in I_n^+$. Let $y \in [0, \infty]$. Obviously,

$$s(x) = s(q) \ge 1 \quad \text{whenever } x \in s^{-1}[\{y\}] \text{ and } q \in (0,1) \cap \mathbb{Q} \cap s^{-1}[\{y\}].$$

It follows that

$$y \geq 1 \quad \text{whenever } (0,1) \cap \mathbf{int}\, s^{-1}[\{y\}] \neq \emptyset$$

since, in this case, $Q \cap s^{-1}[\{y\}] \neq \emptyset$.

Therefore,

$$\begin{aligned} I_n^+(s) &= \sum_{y \in \mathbf{rng}\, s} y||s^{-1}[\{y\}]|| \\ &= \sum_{y \in \mathbf{rng}\, s} y||\mathbf{int}\, s^{-1}[\{y\}]|| \\ &\geq \sum_{y \in \mathbf{rng}\, s} y||(0,1) \cap \mathbf{int}\, s^{-1}[\{y\}]|| \\ &\geq \sum_{y \in \mathbf{rng}\, s} ||(0,1) \cap \mathbf{int}\, s^{-1}[\{y\}]|| \\ &= \sum_{y \in \mathbf{rng}\, s} ||(0,1) \cap s^{-1}[\{y\}]|| \\ &= 1. \end{aligned}$$

Thus

$$\mathbf{r}(1_Q) \geq 1.$$

Let $q : \mathbb{N} \to \mathbb{Q} \cap [0,1]$ be univalent with range $\mathbb{Q} \cap [0,1]$. For each $\nu \in \mathbb{N}$ let

$$s_\nu = \sum_{\mu=0}^{\nu} 1_{\{q_\mu\}} \in \mathcal{S}_n^+.$$

Note that $s \in \mathcal{S}_{n,\uparrow}^+$ is a nondecreasing sequence in $\mathcal{S}_n^+$, that

$$I_1^+(s_\nu) = \sum_{\mu=0}^{\nu} ||\{q_\mu\}|| = 0$$

and that

$$1_Q = \sup s \leq \sup s.$$

Thus

$$\mathbf{l}(1_Q) \leq I_{n,\uparrow}^n(s) = 0.$$

Theorem 1.1. Suppose $A \in \mathcal{M}_n$, B is a nondecreasing sequence in $\mathcal{M}_n$ and $A \subset \cup_{\nu=0}^\infty B_\nu$. Then

$$||A|| \leq \sup_\nu ||B_\nu||.$$

Proof. We define the sequence C in $\mathcal{M}_n$ by letting $C_0 = B_0$ and for each $\nu \in \mathbb{N}^+$ letting $C_\nu = B_\nu \sim B_{\nu-1}$. Then C is disjointed and $B_\nu = \cup_{\mu=0}^\nu C_\mu$ for each $\nu \in \mathbb{N}$.

Suppose $1 < \lambda < \infty$. Choose a compact multirectangle K such that $K \subset A$ and $||A|| \leq \lambda||K||$. For each $\nu \in \mathbb{N}$ choose an open multirectangle U_ν such that $C_\nu \subset U_\nu$ and $||U_\nu|| \leq \lambda||C_\nu||$. Then $K \subset \cup_{\nu=0}^\infty U_\nu$ so there is $N \in \mathbb{N}$ such that $K \subset \cup_{\mu=0}^N U_\mu$. Thus

$$\lambda^{-1}||A|| \leq ||K|| \leq || \cup_{\mu=0}^N U_\mu|| \leq \sum_{\mu=0}^N ||U_\mu|| \leq \lambda \sum_{\mu=0}^N ||C_\mu|| = \lambda||B_N||.$$

Owing to the arbitrariness of λ the Lemma is proved. $\square$

Corollary 1.2. Suppose $A \in \mathcal{M}_n$, $\mathcal{B}$ is a countable subfamily of $\mathcal{M}_n$ and $A \subset \cup\mathcal{B}$. Then

$$||A|| \le \sum_{B\in\mathcal{B}} ||B||.$$

Proof. In case $\mathcal{B}$ is finite this follows from earlier work. So suppose $\mathcal{B}$ is infinite, let B be an enumeration of $\mathcal{B}$ and, for each $\nu \in \mathbb{N}$, let $C_\nu = \cup_{\mu=0}^{\nu} B_\mu$. Then C is a nondecreasing sequence in $\mathcal{M}_n$ whose union contains A so that, by the preceding Theorem,

$$||A|| \le \sup_\nu ||C_\nu|| \le \sup_\nu \sum_{\mu=0}^{\nu} ||B_\mu|| = \sum_{\nu=0}^{\infty} ||B_\nu||.$$

□

Theorem 1.2. We have

$$\mathbf{l}(s) = I_n^+(s) \quad \text{for any } s \in \mathcal{S}_n^+.$$

Proof. Suppose $s \in \mathcal{S}_n^+$. Obviously, $\mathbf{l}(s) \le I_n^+(s)$

Suppose t is a nondecreasing sequence in $\mathcal{S}_n^+$ and $s \le \sup_\nu t_\nu$. Let $Y = (\mathbf{rng}\, s) \sim \{0\}$ and for each $y \in Y$ let $A_y = \{x \in \mathbb{R}^n : s(x) = y\}$. Suppose $0 < \lambda < 1$ and for each $y \in Y$ and $\nu \in \mathbb{N}$ let $B_{y,\nu} = \{x \in A_y : \lambda y < t_\nu(y)\}$. Then $A_y \subset \cup_{\nu=0}^{\infty} B_{y,\nu}$ so $||A_y|| \le \sup_\nu ||B_{y,\nu}||$ by the preceding Theorem. For each $y \in Y$ and $\nu \in \mathbb{N}$ we have $\lambda y 1_{B_{y,\nu}} \le 1_{A_y} t_\nu$ and this implies that

$$\lambda y||A_y|| \le \lambda y \sup_\nu ||B||_\nu = \sup_\nu \lambda y ||B||_\nu \le I_n\left(1_{A_y} t_\nu\right).$$

Thus

$$\begin{aligned}
\lambda I_n(s) &= \sum_{y\in Y} \lambda y ||A_y|| \\
&\le \sum_{y\in Y} \sup_\nu I_n(1_{A_y} t_\nu) \\
&= \sup_\nu I_n\left(\sum_{y\in Y} 1_{A_y} t_\nu\right) \\
&\le \sup I_n(t_\nu)
\end{aligned}$$

Letting $\lambda \uparrow 1$ we find that $I_n(s) \le \sup I_n(t_\nu)$. Thus $I_n(s) \le \mathbf{l}(s)$. □

Corollary 1.3. We have

$$|I_n(s)| \le \mathbf{l}(|s|) \quad \text{for any } s \in \mathcal{S}_n.$$

Proof. This follows from the preceding Theorem since $|I_n(s)| \le I_n^+(|s|)$ for any $s \in \mathcal{S}_n$. □

Definition 1.3. We let

$$\mathbf{Riem}_n$$

be the set of $f \in \mathcal{F}_n$ such that for each $\epsilon > 0$ there is $s \in \mathcal{S}_n$ such that $\mathbf{r}(|f - s|) < \epsilon$. Thus $\mathbf{Riem}_n$ is the closure of $\mathcal{S}_n$ with respect to the extended seminorm$\mathcal{F}_n \ni f \mapsto \mathbf{r}(|f|)$.

We say $f \in \mathcal{F}_n$ is **Riemann integrable** if $f \in \mathbf{Riem}_n$.

We let
$$\mathbf{Leb}_n$$
be the set of $f \in \mathcal{F}_n$ such that for each $\epsilon > 0$ there is $s \in \mathcal{S}_n$ such that $\mathbf{l}(|f-s|) < \epsilon$. Thus $\mathbf{Leb}_n$ is the closure of $\mathcal{S}_n$ with respect to the extended seminorm $\mathcal{F}_n \ni f \mapsto \mathbf{l}(|f|)$ of $\mathcal{S}_n$.

We say $f \in \mathcal{F}_n$ is **Lebesgue integrable** if $f \in \mathbf{Leb}_n$.

Proposition 1.6. Suppose $f \in \mathcal{F}_n$. Then $f \in \mathbf{Riem}_n$ if and only if for each $\epsilon > 0$ there is $g \in \mathbf{Riem}_n$ such that $\mathbf{r}(|f-g|) < \epsilon$ and $f \in \mathbf{Leb}_n$ if and only if for each $\epsilon > 0$ there is $g \in \mathbf{Leb}_n$ such that $\mathbf{r}(|f-g|) < \epsilon$.

Proof. This should be obvious. It boils down to the fact the the closure of the closure equals the closure. □

Theorem 1.3. $\mathbf{Riem}_n$ is a linear subspace of $\mathcal{F}_n$ and there is one and only one linear function
$$\mathbf{R} : \mathbf{Riem}_n \to \mathbb{R}$$
such that
(i) $\mathbf{R}(s) = I_n(s)$ whenever $s \in \mathcal{S}_n$;
(ii) $|\mathbf{R}(f)| \le \mathbf{r}(|f|)$ whenever $f \in \mathbf{Riem}_n$.

$\mathbf{Leb}_n$ is a linear subspace of $\mathcal{F}_n$ and there is one and only one linear function
$$\mathbf{L} : \mathbf{Leb}_n \to \mathbb{R}$$
such that
(i) $\mathbf{L}(s) = I_n(s)$ whenever $s \in \mathcal{S}_n$;
(ii) $|\mathbf{L}(f)| \le \mathbf{l}(|f|)$ whenever $f \in \mathbf{Leb}_n$.

Proof. Keeping in mind Corollaries 1.1 and 1.3 this follows from two applications of the Abstract Closure Principle. □

Remark 1.4. Suppose $f \in \mathbf{Riem}_n$ and $\epsilon > 0$. Choose $s \in \mathcal{S}_n$ such that $\mathbf{r}(f-s) \le \epsilon$. Then
$$|\mathbf{R}(f) - I_n(s)| = |\mathbf{R}(f) - \mathbf{R}(s)| = |\mathbf{R}(f-s)| \le \mathbf{r}(f-s) \le \epsilon.$$
Suppose $f \in \mathbf{Leb}_n$ and $\epsilon > 0$. Choose $s \in \mathcal{S}_n$ such that $\mathbf{l}(f-s) \le \epsilon$. Then
$$|\mathbf{L}(f) - I_n(s)| = |\mathbf{L}(f) - \mathbf{L}(s)| = |\mathbf{L}(f-s)| \le \mathbf{l}(f-s) \le \epsilon.$$

Example 1.2. Let Q be as in the preceding Example. It follows from the foregoing that
$$1_Q \in \mathbf{Leb}_1.$$
I claim that
$$1_Q \notin \mathbf{Riem}_1.$$
Suppose $s \in \mathcal{S}_1$, $m \in \mathcal{S}_1^+$ and $|1_Q - s| \le m$. Suppose $y \in \mathbb{R}$, $z \in [0,\infty)$ and $I = \mathbf{int}\, s^{-1}[\{y\}] \cap m^{-1}[\{z\}]$. Then
$$|1-y| = |1_Q(x) - s(x)| \le m(x) = z \quad \text{if } x \in I \cap (0,1) \cap \mathbb{Q}$$
and
$$|y| = |1_Q(x) - s(x)| \le m(x) = z \quad \text{if } x \in I \cap (0,1) \sim \mathbb{Q}$$
from which it follows that $1/2 \le z$ whenever $x \in I$. Thus $1/2 \le I_1^+(m)$.

Definition 1.4. Suppose $a, b \in \overline{\mathbb{R}}$. We let

$$a \wedge b = \min\{a, b\} \quad \text{and we let} \quad a \vee b = \max\{a, b\}.$$

Note that

$$a \vee b + a \wedge b = a + b \quad \text{whenever } a, b \in \mathbb{R}.$$

For $c \in \overline{\mathbb{R}}$ we let

$$c^+ = c \vee 0 \quad \text{and we let} \quad c^- = -(c \wedge 0)$$

and we note that

$$c = c^+ - c^- \quad \text{and that} \quad |c| = c^+ + c^-.$$

Proposition 1.7. Suppose $f \in \mathcal{F}_n$ and $f \geq 0$. Then

$$f \in \mathbf{Riem}_n \ \Rightarrow \ \mathbf{R}(f) \geq 0 \quad \text{and} \quad f \in \mathbf{Leb}_n \ \Rightarrow \ \mathbf{L}(f) \geq 0.$$

Proof. Suppose $\epsilon > 0$, $s \in \mathcal{S}_n$ and $\mathbf{r}(|f - s|) < \epsilon$. Then $|f - s^+| \leq |f - s|$ so

$$|\mathbf{R}(f) - I_n^+(s^+)| = |\mathbf{R}(f) - \mathbf{R}(s^+)| = |\mathbf{R}(f - s^+)| \leq \mathbf{r}(|f - s|^+) \leq \mathbf{r}(|f - s|) < \epsilon$$

which implies

$$\mathbf{Riem}(f) \geq I_n^+(s^+) - \epsilon \geq -\epsilon.$$

Owing to the arbitrariness of ϵ we infer that $\mathbf{R}(f) \geq 0$.

In the same way one shows that if $f \in \mathbf{Leb}_n$ then $\mathbf{L}(f) \geq 0$. □

Corollary 1.4. We have

$$f, g \in \mathbf{Riem}_n \text{ and } f \leq g \ \Rightarrow \ \mathbf{R}(f) \leq \mathbf{R}(g)$$

and

$$f, g \in \mathbf{Leb}_n \text{ and } f \leq g \ \Rightarrow \ \mathbf{L}(f) \leq \mathbf{L}(g).$$

Proof. Apply the preceding Proposition to $g - f$. □

Theorem 1.4. Suppose $f \in \mathbf{Riem}_n$. Then $f \in \mathbf{Leb}_n$ and

$$\mathbf{R}(f) = \mathbf{L}(f).$$

Exercise 1.1. Prove Theorem 1.4.

Exercise 1.2. Show that the product of two Riemann integrable functions is Riemann integrable.

Theorem 1.5. Suppose $f \in \mathcal{F}_n$, $f \geq 0$ and $0 \leq c < \infty$. Then

$$f \in \mathbf{Riem}_n \ \Rightarrow \ f \wedge c \in \mathbf{Riem}_n$$

and

$$f \in \mathbf{Leb}_n \ \Rightarrow \ f \wedge c \in \mathbf{Leb}_n.$$

Suppose $f, g \in \mathcal{F}_n$. Then

$$f, g \in \mathbf{Riem}_n \ \Rightarrow \ f \wedge g, \ f \vee g \in \mathbf{Riem}_n$$

and

$$f, g \in \mathbf{Leb}_n \ \Rightarrow \ f \wedge g, \ f \vee g \in \mathbf{Leb}_n.$$

Lemma 1.1. Suppose $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned}
|a \wedge b - a \wedge d| &\le |b - d|, \\
|a \wedge b - c \wedge d| &\le |a - c| + |b - d|, \\
|a \vee b - a \vee d| &\le |b - d|, \\
|a \vee b - c \vee d| &\le |a - c| + |b - d|.
\end{aligned}$$

Proof. To prove the first inequality, suppose $b < d$ and consider the three cases $a \le b$, $a < b < d$, $d \le a$; then note that the inequality is symmetric in b and d.

To prove the second inequality note that

$$|a \wedge b - c \wedge d| \le |a \wedge b - a \wedge d| + |a \wedge d - c \wedge d|$$

and then use the first inequality.

One may use the same techniques to prove the third and fourth inequality. $\square$

Exercise 1.3. Prove Theorem 1.5. Make use of the preceding Remark and Lemma.

Definition 1.5. Suppose $A \subset \mathbb{R}^n$ and f is a real valued function whose domain contains A. We let

$$f_A : \mathbb{R}^n \to \mathbb{R}$$

be such that

$$f_A(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{else.} \end{cases}$$

We set

$$\mathbf{R}_A(f) = \mathbf{R}(f_A) \quad \text{if } f_A \in \mathbf{Riem}_n$$

in which case we say f **is Riemann integrable over** A and we set

$$\mathbf{L}_A(f) = \mathbf{L}(f_A) \quad \text{if } f_A \in \mathbf{Leb}_n$$

in which case we say f **is Lebesgue integrable over** A**.** So, for example, if $A = (a, b)$,

$$\int_a^b f(x)\,dx$$

is, by definition, one or both of $\mathbf{R}_{(a,b)}(f)$ or $\mathbf{L}_{(a,b)}(f)$.

Proposition 1.8. Suppose $f \in \mathcal{F}_n$ and $S \in \mathcal{M}_n$. Then

$$f \in \mathbf{Riem}_n \;\Rightarrow\; 1_S f \in \mathbf{Riem}_n$$

and

$$f \in \mathbf{Leb}_n \;\Rightarrow\; 1_S f \in \mathbf{Leb}_n.$$

Exercise 1.4. Prove Proposition 1.8.

2. The theory of the Lebesgue integral.

2.1. **The Monotone Convergence Theorem.** The theory of the Lebesgue integral rest on the following Theorem.

Theorem 2.1. (**The Monotone Convergence Theorem.**) Suppose f is a nondecreasing sequence in $\mathcal{F}_n^+$. Then

$$\mathbf{l}(\sup_\nu f_\nu) = \sup_\nu \mathbf{l}(f_\nu). \tag{1}$$

Proof. Let a and b be the left and right hand sides of (1), respectively. Owing to the monotonicity of $\mathbf{l}$, we find that $b \le a$. Thus we need only show that $a \le b$ and we may assume that $b < \infty$.

To this end, let $\epsilon > 0$. For each $\nu \in \mathbb{N}$ let $s_\nu \in \mathcal{S}_{n,\uparrow}^+$ be such that $f_\nu \le \sup s_\nu$ and

$$I_{n,\uparrow}^n(s_\nu) \le \mathbf{l}(f_\nu) + 2^{-\nu-1}\epsilon.$$

For each $\mu, \nu \in \mathbb{N}$ with $\mu \le \nu$ we let

$$S_\mu^\nu = \bigvee_{\eta=\mu}^{\nu} s_\eta \in I_{n,\uparrow}^n.$$

We define the sequence t by letting

$$t_\nu = (S_0^\nu)_\nu \in \mathcal{S}_n^+.$$

For any $\nu \in \mathbb{N}$ we have

$$t_\nu = (S_0^\nu)_\nu \le (S_0^{\nu+1})_\nu \le (S_0^{\nu+1})_{\nu+1} = t_{\nu+1}$$

so $t \in \mathcal{S}_{n,\uparrow}^+$ and

$$I_n^+(t_\nu) = I_n^+((S_0^\nu)_\nu) \le I_{n,\uparrow}^n(S_0^\nu). \tag{1}$$

Moreover, for any $\nu, \xi \in \mathbb{N}$, we have

$$(s_\nu)_\xi \le (s_\nu)_{\nu\vee\xi} \le (S_0^{\nu\vee\xi})_{\nu\vee\xi} = t_{\nu\vee\xi} \le \sup t;$$

it follows that $f_\nu \le \sup t$ for any $\nu \in \mathbb{N}$ which in turn implies that $\sup f \le \sup t$ so

$$\mathbf{l}(\sup f) \le I_{n,\uparrow}^n(t).$$

We will complete the proof by showing that

$$I_{n,\uparrow}^n(t) \le \sup_\nu \mathbf{l}(f_\nu) + \epsilon. \tag{2}$$

Suppose $\mu, \nu \in \mathbb{N}$ and $\mu < \nu$. Since $s_\mu \le S_\mu^\nu$ we have

$$f_\mu \le f_\nu \wedge f_{\mu+1} \le (\sup s_\nu) \wedge (\sup S_{\mu+1}^\nu) = \sup(s_\nu \wedge S_{\mu+1}^\nu).$$

Using the fact that $a \wedge b + a \vee b = a + b$ whenever $a, b \in [0, \infty]$ we find that

$$s_\mu \wedge S_{\mu+1}^\nu + S_\mu^\nu = s_\mu \wedge S_{\mu+1}^\nu + s_\mu \vee S_{\mu+1}^\nu = s_\mu + S_{\mu+1}^\nu;$$

thus

$$\begin{aligned} \mathbf{l}(f_\mu) + I_{n,\uparrow}^n(S_\mu^\nu) &\le I_{n,\uparrow}^n(s_\mu \wedge S_{\mu+1}^n) + I_{n,\uparrow}^n(S_\mu^\nu) \\ &= I_{n,\uparrow}^n(s_\mu \wedge S_{\mu+1}^n + S_\mu^\nu) \\ &= I_{n,\uparrow}^n(s_\mu + S_{\mu+1}^\nu) \\ &= I_{n,\uparrow}^n(s_\mu) + I_{n,\uparrow}^n(S_{\mu+1}^\nu) \\ &\le \mathbf{l}(f_\mu) + 2^{-\mu-1}\epsilon + I_{n,\uparrow}^n(S_{\mu+1}^\nu). \end{aligned}$$

Since $\mathbf{l}(f_\mu) < \infty$ we obtain
$$I^n_{n,\uparrow}(S^\nu_\mu) \le I^n_{n,\uparrow}(S^\nu_{\mu+1}) + 2^{-\mu-1}\epsilon;$$
Summing from $\mu = 0$ to ν and using (1) we find that
$$I^+_n(t_\nu) \le I^n_{n,\uparrow}(S^\nu_0) \le I^n_{n,\uparrow}(S^\nu_\nu) + \epsilon = I^n_{n,\uparrow}(s_\nu) + \epsilon < \mathbf{l}(f_\nu) + \epsilon^{-\nu-1} + \epsilon$$
thereby establishing (2). □

Corollary 2.1. **(Fatou's Lemma.)** Suppose f is a sequence in $\mathcal{F}^+_n$. Then
$$\mathbf{l}(\liminf_\nu f_\nu) \le \liminf_\nu \mathbf{l}(f_\nu).$$

Proof. For each $\nu \in \mathbb{N}$ let $F_\nu = \inf_{\mu\ge\nu} f_\mu$, note that $\sup_\nu F_\nu = \liminf_\nu f_\nu$ and apply the Monotone Convergence Theorem to F. □

Corollary 2.2. Suppose f is a nondecreasing sequence in $\mathcal{F}^+_n$. Then
$$\mathbf{l}\left(\sum_{\nu=0}^\infty f_\nu\right) \le \sum_{\nu=0}^\infty \mathbf{l}(f_\nu).$$

Proof. We have
$$\mathbf{l}\left(\sum_{\nu=0}^\infty f_\nu\right) = \mathbf{l}\left(\sup_\nu \sum_{\mu=0}^\nu f_\mu\right) = \sup_\nu \mathbf{l}\left(\sum_{\mu=0}^\nu f_\mu\right) \le \sup_\nu \sum_{\mu=0}^\nu \mathbf{l}(f_\mu) = \sum_{\nu=0}^\infty \mathbf{l}(f_\nu).$$
□

2.2. **Basic theory of Lebesgue integration.**

Theorem 2.2. Suppose $f \in \mathcal{F}^+_n \cap \mathbf{Leb}_n$. Then
$$\mathbf{l}(f) = \mathbf{L}(f).$$

Proof. Let $\epsilon > 0$. Choose $s \in \mathcal{S}_n$ such that $\mathbf{l}(|f - s|) < \epsilon/2$. Applying $\mathbf{l}$ to the inequalities $f \le |f - s^+| + s^+$ and $s^+ \le |f - s^+| + f$ we infer that $|\mathbf{l}(f) - \mathbf{l}(s^+)| \le \mathbf{l}(|f - s^+|)$. Also, $|\mathbf{L}(f) - \mathbf{L}(s^+)| \le \mathbf{l}(|f - s^+|)$. Since $\mathbf{l}(s^+) = \mathbf{L}(s^+)$ and since $|f - s^+| \le |f - s|$ we find that $|\mathbf{l}(f) - \mathbf{L}(f)| < \epsilon$ □

Lemma 2.1. Suppose f is a sequence in $\mathcal{F}^+_n \cap \mathbf{Leb}_n$ such that

(i) $\sup_\nu f_\nu(x) < \infty$ for each $x \in \mathbb{R}^n$ and
(ii) $\mathbf{l}(\sup_\nu f_\nu) < \infty$.

Then $\sup_\nu f_\nu \in \mathbf{Leb}_n$.

Proof. Replacing f_ν by $\sup_{0\le\mu\le\nu} f_\mu$ if necessary we may assume without loss of generality that f is nondecreasing.

Let $\epsilon > 0$. Since (ii) holds we may choose $N \in \mathbb{N}$ such that
$$\sup_\nu \mathbf{l}(f_\nu) \le \mathbf{l}(f_N) + \epsilon.$$
It follows from the preceding Proposition that
$$\mathbf{l}(f_\nu - f_N) = \mathbf{L}(f_\nu - f_N) = \mathbf{L}(f_\nu) - \mathbf{L}(f_N) = \mathbf{l}(f_\nu) - \mathbf{l}(f_N)$$
for any $\nu \in \mathbb{N}$ so that
$$\sup_\nu \mathbf{l}(f_\nu - f_N) \le \epsilon.$$

Since f is nondecreasing we may use the Monotone Convergence Theorem to infer that

$$\mathbf{l}((\sup_\nu f_\nu) - f_N) = \mathbf{l}(\sup_\nu (f_\nu - f_N)) = \sup_\nu \mathbf{l}(f_\nu - f_N) \le \epsilon.$$

□

Lemma 2.2. Suppose f is a sequence in $\mathcal{F}_n^+ \cap \mathbf{Leb}_n$. Then $\inf_\nu f_\nu \in \mathbf{Leb}_n$.

Proof. For each $\nu \in \mathbb{N}$ let $F_\nu = \inf_{0 \le \mu \le \nu} f_\mu \in \mathbf{Leb}_n$. Evidently, F is nonincreasing so $\mathbb{N} \ni \nu \mapsto F_0 - F_\nu$ is nondecreasing. Since

$$\inf_\nu F_\nu = F_0 - \sup_\nu (F_0 - F_\nu)$$

and since $\inf_\nu f_\nu = \inf_\nu F_\nu$ this Lemma follows from Lemma 2.2. □

Theorem 2.3. Suppose $F \in \mathcal{F}_n$, $F \ge 0$, $\mathbf{l}(F) < \infty$ and there is a sequence f in $\mathbf{Leb}_n$ such that

$$F(x) = \lim_{\nu \to \infty} f_\nu(x) \quad \text{for } x \in \mathbb{R}^n.$$

Then $F \in \mathbf{Leb}_n$.

Proof. Choose a $s \in \mathcal{S}^+_{n,\uparrow}$ such that $F \le \sup s$ and $I^n_{n,\uparrow}(s) < \infty$. Using Lemmas 2.1 and 2.2 we infer that, for each $\xi \in \mathbb{N}$,

$$F \wedge s_\xi = \inf_\nu \sup_{\mu \ge \nu} f_\mu \wedge s_\xi \in \mathbf{Leb}_n.$$

Since $F = \sup_\xi F \wedge s_\xi$ the Theorem follows from Lemma 2.1. □

Theorem 2.4. (The Lebesgue Dominated Convergence Theorem.) Suppose

(i) f is a sequence in $\mathbf{Leb}_n$ and $F \in \mathcal{F}_n$ is such that

$$\lim_{\nu \to \infty} f_\nu(x) = F(x) \quad \text{for all } x \in \mathbb{R}^n;$$

(ii) g is a sequence in $\mathbf{Leb}_n$ such that

$$|f_\nu| \le g_\nu, \ \nu \in \mathbb{N};$$

(iii) $G \in \mathcal{F}_n^+$,

$$\lim_{\nu \to \infty} g_\nu(x) = G(x) \text{ for all } x \in \mathbb{R}^n \text{ and } \qquad \lim_{\nu \to \infty} \mathbf{l}(g_\nu) = \mathbf{l}(G) < \infty.$$

Then $F \in \mathbf{Leb}_n$ and

$$\lim_{\nu \to \infty} \mathbf{l}(|F - f_\nu|) = 0.$$

In particular,

$$\lim_{\nu \to \infty} \mathbf{L}(f_\nu) = \mathbf{L}(F).$$

Proof. For each $\nu \in \mathbb{N}$ let $h_\nu = G + g_\nu - |F - f_\nu| \in \mathcal{F}_n^+ \cap \mathbf{Leb}_n$. We know from the previous Theorem that G and $|F - f_\nu| = \lim_{\mu \to \infty} |f_\mu - f_\nu|$, $\nu \in \mathbb{N}$ are in $\mathbf{Leb}_n$ Thus, for any $\nu \in \mathbb{N}$,

$$\mathbf{L}(h_\nu) = \mathbf{L}(G) + \mathbf{L}(g_\nu) - \mathbf{L}(|F - f_\nu|)$$

so

$$\mathbf{l}(h_\nu) = \mathbf{l}(G) + \mathbf{l}(g_\nu) - \mathbf{l}(|F - f_\nu|).$$

By Fatou's Lemma we have

$$2\mathbf{l}(G) = \mathbf{l}(\liminf_{\nu\to\infty} h_\nu) \le \liminf_{\nu\to\infty} \mathbf{l}(h_\nu).$$

Since

$$\liminf_{\nu\to\infty} \mathbf{l}(h_\nu) = 2\mathbf{l}(G) - \limsup_{\nu\to\infty} \mathbf{l}(|F - f_\nu|).$$

it follows that

$$\limsup_{\nu\to\infty} \mathbf{l}(|F - f_\nu|) = 0.$$

This in turn implies that $F \in \mathbf{Leb}_n$.

The last conclusion follows from the observation that

$$|\mathbf{L}(F) - \mathbf{L}(f_\nu)| = |\mathbf{L}(F - f_\nu)| \le \mathbf{l}(|F - f_\nu|) \quad \text{for any } \nu \in \mathbb{N}.$$

□

Definition 2.1. We let

$$\mathbf{Leb}_n^+ = \left\{ \sup_\nu f_\nu : f \text{ is a nondecreasing sequence in } \mathcal{F}_n^+ \cap \mathbf{Leb}_n \right\}.$$

Proposition 2.1. Suppose f is a sequence in $\mathbf{Leb}_n^+$. Then $\sup f \in \mathbf{Leb}_n^+$.

Proof. For each $\nu \in \mathbb{N}$ choose a nondecreasing sequence $g_\nu \in \mathbf{Leb}_n^+$ such that $f_\nu = \sup g_\nu$. For each $\nu \in \mathbb{N}$ let

$$h_\nu = \bigvee_{\mu=0}^{\nu} \bigvee_{\xi=0}^{\nu} (g_\mu)_\xi \in \mathbf{Leb}_n^+.$$

Then h is nondecreasing and $f = \sup h$. □

Proposition 2.2. Suppose $f, g \in \mathbf{Leb}_n^+$ and $c \in [0, \infty]$. Then cf, $f + g$, $f \wedge g$ and $f \vee g$ belong to $\mathbf{Leb}_n^+$.

Proof. Straightforward exercise. □

Theorem 2.5. Suppose $f, g \in \mathbf{Leb}_n^+$. Then

$$\mathbf{l}(f + g) = \mathbf{l}(f) + \mathbf{l}(g).$$

Proof. Let p, q be nondecreasing sequences $\mathcal{F}_n^+ \cap \mathbf{Leb}_n$ with suprema f and g, respectively. Using the Monotone Convergence Theorem three times we calculate

$$\begin{aligned}
\mathbf{l}(f + g) &= \sup_\nu \mathbf{l}(p_\nu + q_\nu) \\
&= \sup_\nu \mathbf{L}(p_\nu + q_\nu) \\
&= \sup_\nu \mathbf{L}(p_\nu) + \mathbf{L}(q_\nu) \\
&= \sup_\nu \mathbf{l}(p_\nu) + \mathbf{l}(q_\nu) \\
&= \mathbf{l}(f) + \mathbf{l}(g).
\end{aligned}$$

□

Theorem 2.6. Suppose f is a sequence in $\mathbf{Leb}_n^+$. Then $\sum_{\nu=0}^\infty f_\nu \in \mathbf{Leb}_n^+$ and

$$\mathbf{l}\left(\sum_{\nu=0}^\infty f_\nu \right) = \sum_{\nu=0}^\infty \mathbf{l}(f_\nu).$$

Proof. Since $\sum_{\nu=0}^{\infty} f_\nu = \sup_\nu \sum_{\mu=0}^{\nu} f_\mu$ we infer from Propositions 2.1 and 2.2 that $\sum_{\nu=0}^{\infty} f_\nu \in \mathbf{Leb}_n^+$. Moreover, by the Monotone Convergence Theorem,

$$\mathbf{l}\left(\sum_{\nu=0}^{\infty} f_\nu\right) = \mathbf{l}\left(\sup_\nu \sum_{\mu=0}^{\infty} f_\mu\right) = \sup_\nu \mathbf{l}\left(\sum_{\mu=0}^{\infty} f_\mu\right) = \sup_\nu \left(\sum_{\mu=0}^{\infty} \mathbf{l}(f_\mu)\right) = \sum_{\nu=0}^{\infty} \mathbf{l}(f_\nu).$$

$\square$

2.3. **Lebesgue measure. Sets of measure zero.** This will come in handy.

Definition 2.2. Suppose $A \subset \mathbb{R}^n$. We let

$$\mathbb{Leb}^n(A) = \inf\left\{\sum_{R\in\mathcal{R}} ||R_\nu|| : \mathcal{R} \subset \mathcal{R}_n,\ \mathcal{R} \text{ is countable, and } A \subset \cup\mathcal{R}\right\}$$

and call this nonnegative extended real number the **Lebesgue measure of** A. We say A **has measure zero** if $\mathbb{Leb}^n(A) = 0$.

Proposition 2.3. Suppose $A \subset \mathbb{R}^n$. Then

$$\mathbb{Leb}^n(A) = \inf\{\sup_\nu ||M_\nu|| : M \text{ is a nondecreasing sequence in } \mathcal{M}_n \text{ and } A \subset \cup_{\nu=0}^{\infty} M_\nu\}.$$

Proof. Let a and b be the left and right sides of the equality to be proved.

Suppose R is a sequence in $\mathcal{R}_n$. For each $\nu \in \mathbb{N}$ let $M_\nu = \cup_{\mu=0}^{\nu} R_\mu$. Then M is a nondecreasing sequence in $\mathcal{M}_n$, $\cup_{\nu=0}^{\infty} R_\nu = \cup_{\nu=0}^{\infty} M_n$ and

$$\sup_\nu ||M_\nu|| \le \sup_\nu \sum_{\mu=0}^{\nu} ||R_\mu|| = \sum_{\nu=0} ||R_\nu||.$$

It follows that $b \le a$.

Suppose M is a nondecreasing sequence in $\mathcal{M}_n$. Let $L_0 = M_0$ and for each $\nu \in \mathbb{N}^+$ let $L_\nu = M_\nu \sim M_{\nu-1}$. Then L is a disjointed sequence in $\mathcal{M}_n$ and $\cup_{\nu=0}^{\infty} L_\nu = \cup_{\nu=0}^{\infty} M_\nu$. For each $\nu \in \mathbb{N}$ choose a finite disjointed family $\mathcal{S}_\nu$ of rectangles with union M_ν. Let $\mathcal{R} = \cup_{\nu=0}^{\infty} \mathcal{S}_\nu$. Then $\mathcal{R}$ is a countable family of rectangles with union $\cup_{\nu=0}^{\infty} M_\nu$. It follows that $a \le b$. $\square$

Theorem 2.7. Suppose $A \subset \mathbb{R}^n$. Then $\mathbb{Leb}^n(A) = \mathbf{l}(1_A)$.

Proof. Suppose $t \in \mathcal{S}_{n,\uparrow}^+$ and $1_A \le \sup t$. Suppose $0 < \sigma < \infty$. For each $\nu \in \mathbb{N}$, let $M_\nu = \{t_\nu > \sigma\} \in \mathcal{M}_n$, note that $\sigma 1_{M_\nu} \le 1_{\{t_\nu > \sigma\}}$ so that $\sigma||M_\nu|| \le I_n^+(t_\nu)$. Now $A \subset \cup_{\nu=0}^{\infty} M_\nu$ and M is an increasing sequence in $\mathcal{M}_n$ so

$$\sigma\mathbb{Leb}^n(A) \le \sigma \sup_\nu ||M_\nu|| = \sup_\nu I_\nu(t_\nu) = I_{n,\uparrow}^n(t).$$

Owing to the arbitrariness of σ it follows Proposition 2.3 that $\mathbb{Leb}^n(A) \le \mathbf{l}(1_A)$.

On the other hand, suppose B is a sequence in $\mathcal{R}_n$ such that $A \subset \cup_{\nu=0}^{\infty} B_\nu$. Let t be the sequence such that, for each $\nu \in \mathbb{N}$, $t_\nu = \sum_{\mu=0}^{\nu} 1_{B_\mu}$. Evidently, $t \in \mathcal{S}_{n,\uparrow}^+$ and $1_A \le \sup t$. Thus

$$\mathbf{l}(1_A) \le I_{n,\uparrow}^n(t) = \sum_{\nu=0}^{\infty} I_n^+(1_{B_\nu}) = \sum_{\nu=0}^{\infty} ||B_\nu||.$$

It follows that $\mathbf{l}(1_A) \le \mathbb{Leb}^n(A)$. $\square$

Theorem 2.8. Suppose M is a multirectangle in $\mathbb{R}^n$. Then
$$\mathbb{Leb}^n(M) = ||M||.$$

Proof. Apply Theorem 2.7 and Theorem1.2. $\square$

Proposition 2.4. The following statements hold.

(i) $|\emptyset) = 0$;
(ii) if $A \subset B \subset \mathbb{R}^n$ then $\mathbb{Leb}^n(A) \le \mathbb{Leb}^n(B)$.
(iii) if A is a nondecreasing sequence of subsets of $\mathbb{R}^n$ the
$$\mathbb{Leb}^n(\cup_{\nu=0}^{\infty} A_\nu) = \sup_\nu \mathbb{Leb}^n(A_\nu);$$
(iv) If $\mathcal{A}$ is a countable family of subsets of $\mathbb{R}^n$ then
$$\mathbb{Leb}^n(\cup \mathcal{A}) \le \sum_{A \in \mathcal{A}}^{\infty} \mathbb{Leb}^n(A).$$

Proof. (i) and (ii) are direct consequences of the definition.

Suppose A is a nondecreasing sequence of subsets of $\mathbb{R}^n$. Using Theorem 2.7 and the Monotone Convergence Theorem we find that
$$\mathbb{Leb}^n(\cup_{\nu=0}^{\infty} A_\nu) = \mathbf{l}\left(1_{\cup_{\nu=0}^{\infty} A_\nu}\right) = \sup_\nu \mathbf{l}\left(1_{A_\nu}\right) = \sup_\nu \mathbb{Leb}^n(A_\nu)$$
so (iii) holds.

If A is a sequence of subsets of $\mathbb{R}^n$ then
$$1_{\cup_{\nu=0}^{\infty} A_\nu} = \sup_\nu 1_{\cup_{\mu=0}^{\nu} A_\mu} \le \sum_{\nu=0}^{\infty} 1_{A_\nu}$$
so (iv) follows from Theorem 2.7 and Theorem 2.2. $\square$

Corollary 2.3. Any countable set is a set of measure zero. The union of a countable family of sets of measure zero is a set of measure zero.

Proposition 2.5. If $a \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$ then $\mathbb{Leb}^n(a + A) = \mathbb{Leb}^n(A)$. (That is, outer measure is **translation invariant.**)

Proof. This follows from the corresponding fact for multirectangles. $\square$

2.4. **Nonmeasurable sets.** There exists a countable disjointed family $\mathcal{C}$ of subsets of $\mathbb{R}$ such that

(i) there is $c \in (0, \infty)$ such that $\mathbb{Leb}^n(C) = c$ for $C \in \mathcal{C}$;
(ii) $0 < \mathbb{Leb}^n(\cup \mathcal{C}) < \infty$;

it follows that
$$\mathbb{Leb}^n(\cup \mathcal{C}) < \sum_{C \in \mathcal{C}} \mathbb{Leb}^n(C) = \infty.$$

Remark 2.1. I'm fairly sure this is equivalent to certain forms of the Axiom of Choice. Consult Professor Hodel, the local set theory expert, if you want more information about this.

Remark 2.2. Let C be an enumeration of $\mathcal{C}$ and for each $\nu \in \mathbb{N}$ let $D_\nu = \cup_{\mu=0}^{\nu} C_\mu$. Then for some ν we have
$$\mathbb{Leb}^n(D_\nu \cup C_{\nu+1}) < \mathbb{Leb}^n(D_\nu) + \mathbb{Leb}^n(C_{\nu+1}).$$

Proof. We're going to be terse! Let B be the range of a choice function for
$$\{x + \mathbb{Q} : x \in \mathbb{R}\}.$$
(In the parlance of algebra, B is a set of coset representatives for the reals modulo the rationals.) It follows that
$$\{q + B : q \in \mathbb{Q}\}$$
is a countable partition of $\mathbb{R}$. Now
$$\infty = \mathbb{Leb}^n(\mathbb{R}) \le \sum_{q\in\mathbb{Q}} \mathbb{Leb}^n(q + B) = \sum_{q\in\mathbb{Q}} \mathbb{Leb}^n(B)$$
so
$$\mathbb{Leb}^n(B) > 0.$$
Since
$$\mathbb{Leb}^n(B) \le \sum_{z\in\mathbb{Z}} \mathbb{Leb}^n(B \cap [z, z + 1))$$
there is $z \in \mathbb{Z}$ such that
$$\mathbb{Leb}^n(B \cap [z, z + 1)) > 0.$$
For each $q \in \mathbb{Q}$ let
$$C_q = q + (B \cap [z, z + 1)).$$
Then
$$\mathbb{Leb}^n(C_q) = \mathbb{Leb}^n(C_0) > 0 \quad \text{whenever } q \in \mathbb{Q}.$$
Let
$$\mathcal{C} = \{C_q : q \in \mathbb{Q},\ 0 \le q \le 1\}.$$
Then
$$\mathbb{Leb}^n(\cup\mathcal{C}) \le \mathbb{Leb}^n([z, z + 2)) = 2$$
but
$$\sum_{C\in\mathcal{C}} \mathbb{Leb}^n(C) = \infty\mathbb{Leb}^n(C_0) = \infty.$$
□

2.5. **Lebesgue measurable sets and functions.** To avoid the situation we encountered in the preceding subsection we define a very useful class of set on which $\mathbb{Leb}^n$ behaves very well.

Definition 2.3. We say a subset E of $\mathbb{R}^n$ is **Lebesgue measurable** if for each $\epsilon > 0$ and each bounded rectangle R in $\mathbb{R}^n$ there is a multirectangle M such that $M \subset R$ and
$$\mathbb{Leb}^n(R \cap ((E \sim M) \cup (M \sim E))) < \epsilon.$$
We let
$$\mathcal{L}_n$$
be the family of Lebesgue measurable sets.

Theorem 2.9. Suppose $E \subset \mathbb{R}^n$. Then
$$E \in \mathcal{L}_n \ \Leftrightarrow\ 1_E \in \mathbf{Leb}_n^+.$$

Proof. For each $\nu \in \mathbb{N}$ let $R_\nu = \cap_{i=1}^n \{x \in \mathbb{R}^n : |x_i| < \nu\} \in \mathcal{R}_n$.

Suppose $E \in \mathcal{L}_n$, $\nu \in \mathbb{N}$ and $\epsilon > 0$. Choose $M \in \mathcal{M}_n$ such that $M \subset R_\nu$ and

$$\mathbb{Leb}^n(R_\nu \cap ((E \sim M) \cup (M \sim E))) < \epsilon.$$

Then

$$|1_{R_\nu \cap E} - 1_{M_\nu}| = 1_{R_\nu \cap ((E \sim M) \cup (M \sim E))}$$

so

$$\mathbf{l}\,(1_{R_\nu \cap E} - 1_{M_\nu}) \le \mathbf{l}\left(1_{R_\nu \cap ((E \sim M) \cup (M \sim E))}\right) = \mathbb{Leb}^n(R_\nu \cap ((E \sim M) \cup (M \sim E))) < \epsilon.$$

Owing to the arbitrariness of ϵ we infer that $1_{R_\nu \cap E} \in \mathbf{Leb}_n$. Since $1_{R_\nu \cap E} \uparrow 1_E$ as $\nu \uparrow \infty$ we infer that $1_E \in \mathbf{Leb}_n^+$.

Suppose $1_E \in \mathbf{Leb}_n^+$. Let f be a nondecreasing sequence in $\mathbf{Leb}_n$ such that $f \ge 0$ and $\sup f = 1_E$. Suppose R is a bounded rectangle in $\mathbb{R}^n$. Then $1_R f$ is a nondecreasing sequence in $\mathbf{Leb}_n$ such that $1_R f \ge 0$ and $\sup 1_R f = 1_R 1_E = 1_{R \cap E}$. Let $\epsilon > 0$. By the Mononone Convergence Theorem there is $N \in \mathbb{N}$ such that $\mathbf{l}\,(1_{R\cap E} - 1_R f_N) < \epsilon/4$. Choose $s \in \mathcal{S}_n$ such that $\mathbf{l}\,(1_R f_N - s) < \epsilon/4$ and let $M = R \cap \{s \ge 1/2\} \in \mathcal{M}_n$. Then

$$\begin{aligned}
\frac{1}{2} 1_{R \cap ((E \sim M) \cup (M \sim E))} &= \frac{1}{2}\left|1_{R \cap (E \sim M)} - 1_{R \cap (M \sim E)}\right| \\
&\le |s - 1_{R \cap E}| \\
&\le |s - 1_R f_N| + |1_R f_N - 1_{R \cap E}|;
\end{aligned}$$

it follows that

$$\mathbb{Leb}^n(R \cap ((E \sim M) \cup (M \sim E))) = \mathbf{l}\left(1_{R \cap ((E \sim M) \cup (M \sim E))}\right) < \epsilon$$

so that E is Lebesgue measurable.

□

Theorem 2.10. The following statements hold.

(i) $\mathcal{M}_n \subset \mathcal{L}_n$.

(ii) $E \in \mathcal{L}_n$ and $\mathbb{Leb}^n(E) < \infty$ if and only if for each $\epsilon > 0$ there is a bounded multirectangle M such that

$$\mathbb{Leb}^n((E \sim M) \cup (M \sim E)) \le \epsilon.$$

(iii) $E \in \mathcal{L}_n$ if and only if there is nondecreasing sequence F in $\{G \in \mathcal{L}_n : \mathbb{Leb}^n(G) < \infty\}$ such that $E = \cup_{\nu=0}^\infty F_\nu$.

(iv) If $E, F \in \mathcal{L}_n$ then $E \cup F, E \cap F, E \sim F \in \mathcal{L}_n$ and

$$\mathbb{Leb}^n(E \cup F) + \mathbb{Leb}^n(E \cap F) = \mathbb{Leb}^n(E) + \mathbb{Leb}^n(F).$$

If $\mathcal{E}$ is a countable nonempty family of Lebesgue measurable subsets of $\mathbb{R}^n$ the following assertions hold:

(v) $\cup\mathcal{E}$ and $\cap\mathcal{E}$ are Lebesgue measurable;

(vi) if $\mathcal{E}$ is disjointed then

$$\mathbb{Leb}^n(\cup\mathcal{E}) = \sum_{E \in \mathcal{E}} \mathbb{Leb}^n(E);$$

(vii) if $\mathcal{E}$ is nested then

$$\mathbb{Leb}^n(\cup\mathcal{E}) = \sup\{\mathbb{Leb}^n(E) : E \in \mathcal{E}\};$$

(viii) if $\mathcal{E}$ is nested and $\mathbb{Leb}^n(E) < \infty$ for some $E \in \mathcal{E}$ then
$$\mathbb{Leb}^n(\cap\mathcal{E}) = \inf\{\mathbb{Leb}^n(E) : E \in \mathcal{E}\}.$$

Proof. Exercise for the reader. $\square$

Remark 2.3. In particular, the Lebesgue measurable subsets of $\mathbf{R}^n$ form a σ-*algebra of subsets of* $\mathbb{R}^n$.

Proposition 2.6. Suppose $E \subset \mathbb{R}^n$. Then
$$\mathbb{Leb}^n(E) = \inf\{\mathbb{Leb}^n(G) : G \text{ is open and } E \subset G\}.$$

Proof. Exercise for the reader. This is a straightforward consequence of the definition of $\mathbb{Leb}^n$. $\square$

Theorem 2.11. Suppose $E \subset \mathbb{R}^n$ and $\mathbb{Leb}^n(E) < \infty$. Then E is Lebesgue measurable if and only if
$$\mathbb{Leb}^n(E) = \sup\{\mathbb{Leb}^n(K) : K \text{ is compact and } K \subset E\}.$$

Proof. Exercise for the reader. Here's a start. First reduce to the case when E is bounded. Next, given $\epsilon > 0$ choose a bounded open subset G such that $E \subset G$ and $\mathbb{Leb}^n(E) \le \mathbb{Leb}^n(G) + \epsilon$. Now consider $E \sim G$. $\square$

Definition 2.4. Suppose $f : \mathbb{R}^n \to \mathbb{R}$. We say f is **Lebesgue measurable** if $f^{-1}[U] \in \mathcal{L}_n$ whenever U is an open subset $\mathbb{R}^n$

Proposition 2.7. Suppose $f : \mathbb{R}^n \to \mathbb{R}$. The following are equivalent.

(i) f is Lebesgue measurable.
(ii) $\{x \in \mathbb{R}^n : f(x) > c\} \in \mathcal{L}_n$ whenever $c \in \mathbb{R}$.
(iii) $\{x \in \mathbb{R}^n : f(x) \ge c\} \in \mathcal{L}_n$ whenever $c \in \mathbb{R}$.
(iv) $\{x \in \mathbb{R}^n : f(x) < c\} \in \mathcal{L}_n$ whenever $c \in \mathbb{R}$.
(v) $\{x \in \mathbb{R}^n : f(x) \le c\} \in \mathcal{L}_n$ whenever $c \in \mathbb{R}$.

Proof. Since
$$\{x \in \mathbb{R}^n : f(x) \ge c\} = \cap_{\nu=1}^{\infty} \left\{x \in \mathbb{R}^n : f(x) > c - \frac{1}{\nu}\right\}$$
we see that that (ii) implies (iii). Since
$$\{x \in \mathbb{R}^n : f(x) < c\} = \mathbb{R}^n \sim \{x \in \mathbb{R}^n : f(x) \ge c\}$$
we see that (iii) implies (iv). Since
$$\{x \in \mathbb{R}^n : f(x) \le c\} = \cap_{\nu=1}^{\infty} \left\{x \in \mathbb{R}^n : f(x) < c + \frac{1}{\nu}\right\}$$
we see that (iv) implies (v). Since
$$\{x \in \mathbb{R}^n : f(x) > c\} = \mathbb{R}^n \sim \{x \in \mathbb{R}^n : f(x) \le c\}$$
we see that (v) implies (ii). Thus (ii),(iii),(iv) and (v) are equivalent.

(i) obviously implies (ii). Suppose (ii) holds. Then, as (iv) holds,
$$\{x \in \mathbb{R}^n : a < f(x) < b\} \in \mathcal{L}_n \quad \text{whenever } -\infty < a < b < \infty.$$
Let U be an open subset of $\mathbb{R}$. Let $\mathcal{I}$ be the family of open subintervals of U with rational endpoints. Then, as $\mathcal{I}$ is countable, we find that
$$f^{-1}[U] = \cup\{f^{-1}[I] : I \in \mathcal{I}\} \in \mathcal{L}_n$$

Thus (i) holds. $\square$

Corollary 2.4. Suppose N is a positive integer, $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, N$ are Lebesgue measurable functions, and

$$M : \mathbb{R}^N \to \mathbb{R}$$

is continuous. Then

$$\mathbb{R}^n \ni x \mapsto M(f_1(x), \ldots, f_N(x))$$

is Lebesgue measurable.

Corollary 2.5. The set of Lebesgue measurable functions is closed under the arithmetic operation as well as the lattice operations.

Proposition 2.8. Suppose f is a sequence of Lebesgue measurable functions and $F : \mathbb{R}^n \to \mathbb{R}$ is such that

$$\lim_{\nu \to \infty} f_\nu(x) = F(x) \quad \text{whenever } x \in \mathbb{R}^n.$$

Then F is Lebesgue measurable.

Proof. Suppose $c \in \mathbb{R}$. Then

$$\{x \in \mathbb{R}^n : F(x) > c\} = \cup_{n=1}^\infty \cup_{N=0}^\infty \cap_{\nu=N}^\infty \left\{x \in \mathbb{R}^n : f_\nu(x) > c + \frac{1}{n}\right\}.$$

$\square$

Lemma 2.3. Suppose $f : \mathbb{R}^n \to \mathbb{R}$, $c \in \mathbb{R}$, $E = \{x \in \mathbb{R}^n : f(x) > c\}$ and

$$g_h(x) = \frac{1}{h}[f \wedge (c + h) - f \wedge c] \quad \text{for } h \in (0, \infty).$$

Then

(i) $g_h \leq g_k$ if $0 < k < h < \infty$;
(ii) $1_E = \sup_{0<h<\infty} g_h$.

Proof. To prove (i) we suppose $a \in \mathbb{R}^n$ and $0 < k < h < \infty$ and we observe that

$$\begin{aligned}
f(a) < c \ &\Rightarrow\ g_h(a) = 0 = g_k(a), \\
c \leq f(a) < c + k \ &\Rightarrow\ g_h(a) = \frac{1}{h}[f(a) - c] \leq \frac{1}{k}[f(a) - c] = g_k(a), \\
c + k \leq f(a) < c + h \ &\Rightarrow\ g_h(a) = \frac{1}{h}[f(a) - c] \leq 1 = g_k(a), \\
c + h \leq f(a) \ &\Rightarrow\ g_h(a) = 1 = g_k(a).
\end{aligned}$$

(ii) is evident. $\square$

Lemma 2.4. Suppose

$$f : \mathbb{R}^n \to \mathbb{R};$$

c is a a sequence of positive real numbers such that

$$\lim_{\nu \to \infty} c_\nu = 0 \quad \text{and} \quad \sum_{\nu=0}^\infty c_\nu = \infty;$$

and E is the sequence of subsets of $\mathbb{R}^n$ defined inductively by setting $E_0 = \{x \in \mathbb{R}^n : f(x) > c_0\}$ and requiring that

$$E_{\nu+1} = \left\{ x \in \mathbb{R}^n : f(x) > \sum_{\mu=0}^{\nu} c_\mu 1_{E_\mu} \right\} \quad \text{whenever } \nu > 0.$$

Then

$$f = \sum_{\nu=0}^{\infty} c_\nu 1_{E_\nu}.$$

Proof. Straightforward exercise. □

Theorem 2.12. Suppose $f : \mathbb{R}^n \to \mathbb{R}$. Then $f \in \mathbf{Leb}_n$ if and only if $\mathbf{l}(f) < \infty$ and f is Lebesgue measurable.

Proof. Suppose $f \in \mathbf{Leb}_n$. Let $c \in \mathbb{R}$. That $\{x \in \mathbb{R}^n : f(x) > c\} \in \mathcal{L}_n$ follows the first of the two preceding Lemmas and our earlier theory.

Suppose $\mathbf{l}(f) < \infty$ and f is Lebesgue measurable. Writing $f = f^+ - f^-$ we see we need only consider the case $f \geq 0$. Let c be a sequence of positive real numbers such that $\lim_{\nu\to\infty} c_\nu = 0$ and $\sum_{\nu=0}^{\infty} c_\nu = \infty$ and let the sequence E be as in the preceding Lemma so that

$$f = \sum_{\nu=0}^{\infty} c_\nu 1_{E_\nu}.$$

Note that $E_\nu \in \mathcal{L}_n$. That $f \in \mathbf{Leb}_n$ follows from earlier theory. □

Theorem 2.13. (The absolute continuity of the integral.) Suppose $f \in \mathbf{Leb}_n$. Then for each $\epsilon > 0$ there is $\delta > 0$ such that

$$E \in \mathcal{L}_n \text{ and } |E| < \delta \;\Rightarrow\; \mathbf{L}_E(|f|) < \epsilon.$$

Proof. For each nonnegative integer ν let $g_\nu = |f| \wedge \nu$. Since $g_\nu \uparrow |f|$ as $\nu \uparrow \infty$ we infer from the Monotone Convergence Theorem that $\mathbf{l}(g_\nu) \uparrow \mathbf{l}(|f|)$ as $\nu \uparrow \infty$. Choose a positive integer N such that

$$\mathbf{l}(|f|) - \mathbf{l}(g_N) < \frac{\epsilon}{2}.$$

By the preceding theory, $g_N \in \mathbf{Leb}_n$. Let $\delta = \frac{\epsilon}{2N}$. If $E \in \mathcal{L}_n$ and $|E| < \delta$ then

$$|f|1_E = (|f| - g_N)1_E + g_N 1_E \leq |f| - g_N + N1_E$$

so that

$$\mathbf{L}_E(|f|) = \mathbf{L}(|f|1_E) \leq \mathbf{L}(|f| - g_N + N1_E) = \mathbf{L}(|f|) - \mathbf{L}(g_N) + N|E| < \epsilon.$$

□

3. More on the Riemann integral.

The Riemann integral isn't so great but everybody studies it because it's easier to define.

The Definition we gave of the Riemann integral is not the standard one. Now we show that it is equivalent to the standard one.

Definition 3.1. Suppose $f \in \mathcal{B}_n$ and $\delta > 0$. We let

$$\mathbf{RiemSum}_\delta(f)$$

be the set of sums

$$\sum_{R \in \mathcal{R}} f(c(R)||R||$$

where

(i) $\mathcal{R}$ is a finite nonoverlapping family of nonempty bounded rectangles;
(ii) $\mathbf{diam}\, R < \delta$ whenever $R \in \mathcal{R}$;
(iii) $\{f \neq 0\} \subset \cup\mathcal{R}$;
(iv) c is a choice function for $\mathcal{R}$.

The members of $\mathbf{RiemSum}_\delta(f)$ are called **Riemann sums for** f **with mesh diameter at most** δ**.**

Theorem 3.1. Suppose $f \in \mathcal{B}_n$. Then $f \in \mathbf{Riem}_n$ if and only if

$$\inf\{\mathbf{diam}\, \mathbf{RiemSum}_\delta(f) : \delta > 0\} = 0$$

in which case

$$\cap_{0<\delta<\infty} \mathbf{RiemSum}_\delta(f) = \{R(f)\}.$$

Proof. For each $\nu \in \mathbb{N}^+$ let $C_\nu = \{x \in \mathbb{R}^n : |x_i| < \nu\}$. Let N be the least $\nu \in \mathbb{N}$ such that $\{x \in \mathbb{R}^n : f(x) \neq 0\} \subset C_\nu$.

Part One. Suppose $\inf\{\mathbf{diam}\, \mathbf{RiemSum}_\delta(f) : 0 < \delta < \infty\} = 0$**.** Let $\delta > 0$ and let $\mathcal{R}$ be a finite disjointed family of nonempty rectangles such that $C_N = \cup\mathcal{R}$ and $\mathbf{diam}\, R < \delta$ for $R \in \mathcal{R}$.

Let $\underline{c}$ and $\overline{c}$ be choice functions for $\mathcal{R}$ such that

$$f(\underline{c}(R)) \leq \inf_R f + \delta \quad \text{and} \quad \sup_R f \leq f(\overline{c}(R)) + \delta \quad \text{whenever } R \in \mathcal{R}.$$

Let

$$\underline{S} = \sum_{R \in \mathcal{R}} f(\underline{c}(R))||R||; \quad \text{let} \quad \overline{S} = \sum_{R \in \mathcal{R}} f(\overline{c}(R))||R||;$$

let

$$s = \sum_{R \in \mathcal{R}} (\inf_R f) 1_R \in \mathcal{S}_n; \quad \text{and let} \quad m = \sum_{R \in \mathcal{R}} (\sup_R f - \inf_R f) 1_R \in \mathcal{S}_n^+.$$

Then $|f - s| \leq m$ and, since $\underline{S}, \overline{S} \in \mathbf{RiemSum}_\delta(f)$, we find that

$$\begin{aligned} I(m) &= \sum_{R \in \mathcal{R}} (\sup_R f - \inf_R f)||R|| \\ &\leq \sum_{R \in \mathcal{R}} (f(\overline{c}(R)) - f(\underline{c}(R)) + 2\delta||R|| \\ &\leq \overline{S} - \underline{S} + 2\delta||C_N|| \\ &\leq \mathbf{diam}\, \mathbf{RiemSum}_\delta(f) + 2\delta||C_N||. \end{aligned}$$

Owing to the arbitrariness of δ it follows that $f \in \mathbf{Riem}_n$.

Part Two. Suppose $f \in \mathbf{Riem}_n$ **and** $\epsilon > 0$**.** Choose $s \in \mathcal{S}_n$, $m \in \mathcal{S}_n^+$ such that $|f - s| \leq m$, $I_n^+(m) < \epsilon/4$. We will show that there is $\delta > 0$ such that if $\Sigma \in \mathbf{Riemsum}_\delta(f)$ then $|\Sigma - I(s)| < \epsilon/2$; that will imply that $\mathbf{diam}\, \mathbf{Riemsum}_\delta(f) < \epsilon$. Since $|f - 1_{C_{N+1}} s| \leq 1_{C_{N+1}} m$ we may assume that $\{x \in \mathbb{R}^n : s(x) \neq 0$ or $m(x) \neq$

$0\} \subset C_{N+1}$. It follows from ?? that there is finite disjointed family of rectangles $\mathcal{Q}$ and functions $\sigma : \mathcal{Q} \to \mathbb{R}$ and $\mu : \mathcal{Q} \to [0, \infty)$ such that

$$s = \sum_{Q \in \mathcal{Q}} \sigma(Q) 1_Q \quad \text{and} \quad m = \sum_{Q \in \mathcal{Q}} \mu(Q) 1_Q.$$

Suppose $0 < \delta < 1$ and $\mathcal{R}$ and c are as in Definition ??. We may assume that $\cup\mathcal{R} \subset C_{N+1}$ since if $\mathcal{R}' = \{R \in \mathcal{R}\colon\ R \sim C_{N+1} \neq \emptyset\}$ then, as $\delta < 1$, $\cup\mathcal{R}' \subset \mathbb{R}^n \sim C_N$ so $f(c(R)) = 0$ for $R \in \mathcal{R}'$ and $\sum_{R \in \mathcal{R}} f(c(R))||R|| = \sum_{R \in \mathcal{R} \sim \mathcal{R}'} f(c(R))||R||$. We may also assume that $\cup\mathcal{R} = C_{N+1}$ ... Let

$$\mathcal{G} = \{(Q, R) \in \mathcal{Q} \times \mathcal{R} : c(R) \in Q\} \quad \text{and let} \quad \mathcal{B} = \{(Q, R) \in \mathcal{Q} \times \mathcal{R} : c(R) \notin Q\}.$$

Then

$$\begin{aligned} |I_n(s) - \Sigma| &\leq \sum_{(Q,R) \in \mathcal{Q} \times \mathcal{R}} |\sigma(Q) - f(c(R))||Q \cap R|| \\ &\leq \sum_{(Q,R) \in \mathcal{G}} \mu(Q)||Q \cap R|| + \sum_{(Q,R) \in \mathcal{B}} M||Q \cap R|| \\ &\leq I(m) + M \sum_{(Q,R) \in \mathcal{B}} ||Q \cap R||. \end{aligned}$$

Now if $(Q, R) \in \mathcal{B}$ and $||Q \cap R|| \neq 0$ then R is contained in the $\sqrt{n}\delta$ neighborhood of $\mathbf{bdry}\, Q$ $\square$

3.1. **The fundamental theorems of calculus.**

Theorem 3.2. Suppose $-\infty < a < b < \infty$, $f : [a, b] \to \overline{\mathbb{R}}$, f is differentiable at each point of (a, b) and f' is Riemann integrable on (a, b). Then

$$\mathbf{R}_{(a,b)}(f')\, dx = f(b) - f(a). \tag{3}$$

Remark 3.1. Using more traditional notation, (6) says

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

Remark 3.2. Suppose $-\infty < a < b < \infty$, $f : (a, b) \to \overline{\mathbb{R}}$, f is differentiable at each point of (a, b) and f' is Riemann integrable on (a, b). Then there is $M \in [0, \infty)$ such that $|f'(x)| \leq M$ whenever $a < x < b$. This implies $|f(x) - f(y)| \leq M|x - y|$ whenever $a < x < y < b$ which is to say that $\mathbf{Lip}\, f \leq M$. In particular, f has a unique continuous extension to the closure $[a, b]$ of (a, b).

Exercise 3.1. Prove Theorem 3.2. Note that

$$f(b) - f(a) = \sum_{i=1}^{N} f(x_i) - f(x_{i-1})$$

whenever $N \in \mathbb{N}^+$ and $a = x_0 \leq x_1 \leq \cdots \leq x_N = b$. Use the Mean Value Theorem to construct Riemann sums which do the job.

Theorem 3.3. Suppose $f : (a, b) \to \overline{\mathbb{R}}$, f is Riemann integrable and

$$F(x) = \mathbf{R}_{(a,x)}(f) \quad \text{for } x \in (a, b).$$

Then

$$F'(x) = f(x) \quad \text{whenever } x \in (a, b) \text{ and } f \text{ is continuous at } x. \tag{4}$$

Remark 3.3. Using more traditional notation, (6) says

$$\frac{d}{dx}\left(\int_a^x f'(t)\,dt\right) = f(x).$$

Exercise 3.2. Prove Theorem 3.3. Don't hesitate to use the theory already developed.

Exercise 3.3. Suppose $1 < p < \infty$. Let $f, g \in \mathcal{F}_n^+$ be such that

$$f(x) = \begin{cases} \frac{1}{x^p} & \text{if } 0 < x < 1, \\ 0 & \text{else} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{1}{x^p} & \text{if } 1 < x < \infty, \\ 0 & \text{else.} \end{cases}$$

Show that $\mathbf{l}(f) < \infty$ if and only if $p < 1$ and show that $\mathbf{l}^+(g) < \infty$ if and only if $p > 1$.

(Big Hint: Use Theorem 3.2 together with the Mononotone Convergence Theorem of the next set of notes.)

3.2. **Characterization of Riemann integrability.** The following Theorem characterizes $\mathbf{Riem}_n$ in a very precise way.

Theorem 3.4. Suppose $f \in \mathcal{B}_n$. Then $f \in \mathbf{Riem}_n$ if and only if the set of discontinuities of f has measure zero.

We will now prove this Theorem. So suppose $f \in \mathcal{B}_n$. Let $M \in [0, \infty)$ be such that $|f| \leq M$ and let S be a compact rectangle such that $\{x \in \mathbb{R}^n : f(x) \neq 0\} \subset S$. For each positive integer ν let

$$D_\nu = \{x \in \mathbb{R}^n : \mathbf{osc} f(x) \geq 1/\nu\}$$

and let $E = \cup_{\nu=1}^\infty E_\nu$. Then E is the set of discontinuities of f.

Suppose $\nu \in \mathbb{N}^+$. By an earlier exercise about $\mathbf{osc} f$, D_ν is closed. Since $D_\nu \subset \{f \neq 0\}$ we find that D_ν is bounded. Thus D_ν is compact.

Lemma 3.1. Suppose $f \in \mathbf{Riem}_n$. There is a disjointed family $\mathcal{R}$ of rectangles such that $\cup\mathcal{R} = S$ and

$$\sum_{R \in \mathcal{R}} (\sup f[R] - \inf f[R])||R|| \leq \epsilon.$$

Proof. Let $s \in \mathcal{S}_n$ and $m \in \mathcal{S}_n^+$ be such that $|f - s| \leq m$ and $I_n^+(m) \leq \epsilon/2$. Replacing s and m by $1_S s$ and $1_S m$ if necessary we may assume without loss of generality that $\{s \neq 0\} \cup \{m > 0\} \subset S$. Choose $\mathcal{R}, \sigma, \mu$ such that $\mathcal{R}$ is a finite disjointed family of rectangles with union S; σ, μ are functions with domain $\mathcal{R}$ and ranges contained in $\overline{\mathbb{R}}$ and $[0, \infty)$, respectively;

$$s = \sum_{R \in \mathcal{R}} \sigma(R) 1_R \quad \text{and} \quad m = \sum_{R \in \mathcal{R}} \mu(R) 1_R.$$

Suppose $R \in \mathcal{R}$. Then

$$\sigma(R) - \mu(R) = s(x) - m(x) \leq f(x) \leq s(x) + m(x) = \sigma(R) + \mu(R) \quad \text{for } x \in R.$$

This implies

$$\sigma(R) - \mu(R) \leq \inf_R f \quad \text{and} \quad \sup_R f \leq \sigma(R) + \mu(R)$$

so

$$(\sup_R f - \inf_R f)||R|| \leq 2\mu(R)||R||.$$

Now sum over $\mathcal{R}$. □

Corollary 3.1. Suppose $f \in \mathbf{Riem}_n$. Then the set of discontinuities of f has measure zero.

Proof. Since $\mathbb{Leb}^n(E) = \mathbb{Leb}^n(\cup_{\nu=1}^\infty D_\nu) \le \sum_{\nu=1}^\infty \mathbb{Leb}^n(D_\nu)$ it will suffice to show that $\mathbb{Leb}^n(D_\nu) = 0$ for all $\nu \in \mathbb{N}^+$.

So suppose $\nu \in \mathbb{N}^+$ and let $\epsilon > 0$. Let $\mathcal{R}$ be as in the preceding Theorem with ϵ there replaced by ϵ/ν. I claim that

$$(5) \qquad \frac{1}{\nu}\mathbb{Leb}^n(D_\nu \cap R) \le (\sup_R f - \inf_R f)||R|| \quad \text{whenever } R \in \mathcal{R}.$$

Suppose $R \in \mathcal{R}$. If $x \in D_\nu \cap \mathbf{int}\, R \neq \emptyset$ we have $1/\nu \le \mathbf{osc} f(x) \le \sup_R f - \inf_R$; moreover, $\mathbb{Leb}^n(D_\nu \cap R \mathbb{Leb}^n(^* \le \mathbb{Leb}^n(R) = ||R||$. If $D_\nu \cap \mathbf{int}\, R$ is empty then $\mathbb{Leb}^n(D_\nu \cap R) \le \mathbb{Leb}^n(\mathbf{bdry}\, R) = ||\mathbf{bdry}\, R|| = 0$. Thus (8) holds. It follows that

$$\mathbb{Leb}^n(D_\nu) \le \sum_{R \in \mathcal{R}} \mathbb{Leb}^n(D_\nu \cap R) \le \nu \sum_{R \in \mathcal{R}} (\sup_R f - \inf_R f)||R|| < \epsilon.$$

Owing to the arbitrariness of ϵ we conclude that $\mathbb{Leb}^n(D_\nu) = 0$. □

Lemma 3.2. Suppose $f \in \mathcal{B}_n$ and the set of discontinuities of f has measure zero. Then $f \in \mathbf{Riem}_n$.

Proof. Suppose $\epsilon > 0$. Choose $\eta > 0$ and $\nu \in \mathbb{N}^+$ such that $||S||/\nu + M\eta < \epsilon$. Let $\mathcal{Z}$ be a countable family of open rectangles such that $D_\nu \subset \cup\mathcal{Z}$ and $\sum_{r \in \mathcal{Z}} ||R|| < \eta$. Since D_ν is compact there is a finite subfamily $\mathcal{F}$ of $\mathcal{Z}$ such that $D_\nu \subset \cup\mathcal{F}$. Let $\delta > 0$ be such that δ is less than the Lebesgue number of the covering

$$\left\{U : \text{is an open subset of } \mathbb{R}^n \text{ and } (\sup_U f - \inf_U f) < \tfrac{1}{\nu}\right\}.$$

of the of the compact set $K = S \sim \cup\mathcal{F}$. Let $\mathcal{R}$ be a finite disjointed family of rectangles with union K none of whose diameters exceed δ; let

$$s = \sum_{S \in \mathcal{R}} \inf_R f 1_R \quad \text{and let} \quad m = \sum_{R \in \mathcal{R}} (\sup_R f - \inf_R f) 1_R + M \sum_{R \in \mathcal{F}} 1_R.$$

Then

$$|f - s| \le m \quad \text{and} \quad I_n^+(m) \le \frac{1}{\nu}||S|| + M\eta < \epsilon.$$

□

Definition 3.2. We say a subset A of $\mathbb{R}^n$ **has Jordan content** if $1_A \in \mathbf{Riem}_n$ in which case we let $\mathbf{R}(1_A)$ be the **Jordan content of** A. In view of the preceding Theorem, A will have Jordan content if and only if A is bounded and its boundary has measure zero. Since the boundary of such a set is compact we find that A has Jordan content if and only if A is bounded and for every $\epsilon > 0$ there is a finite family $\mathcal{R}$ of open rectangles such that $\sum_{R \in \mathcal{R}} ||R|| \le \epsilon$. Since $\mathbf{R}$ is linear Jordan content is additive and if A, B have Jordan content then so do $A \cup B$, $A \cap B$ and $A \sim B$.