How elementary linear maps change areas.

Fix an integer $n \geq 2$. Let

$$C = \{x \in^R n : 0 \leq x_i \leq 1,\ i = 1, \ldots, n\}.$$

The "C" here stands for **cube**. Our goal in this Introduction is to prove that

$$|L[C]| = |\mathbf{det}\, L|, \quad L \in \mathbf{GL}(^R n). \tag{1}$$

For each $c \in \mathbf{R} \sim \{0\}$ let $S_c \in \mathbf{GL}(^R n)$ be defined by

$$S_c(x) = (x_1, x_2, \ldots, x_n + c x_{n-1}), \quad x \in^R n.$$

The 'S' here stands for **shear**; that this is reasonable terminology can be seen by drawing a picture of what S_c does to the cube C. For each $c > 0$ let $D_c \in \underline{\mathrm{GL}}(^R n)$ be defined by

$$D_c(x) = (x_1, x_2, \ldots, c x_n), \quad x \in^R n.$$

The 'D' here stands for **dilate**. For each $\sigma \in^R n$ let $P_\sigma \in \underline{\mathrm{L}}(^R n, ^R n)$ be defined by

$$P_\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad x \in^R n.$$

The 'P' here stands for **permutation**.

It follows from the the Gaussian elimination algorithm from elementary linear algebra that any member of $\mathbf{GL}(^R n)$ can be written as a product of shears, dilations and permutations. **Thus, if we can show that (1) holds for shears, permutations and dilations** we will, in view of the product rule for determinants, have shown that (1) holds for any $L \in \mathbf{GL}(^R n)$.

Suppose $c \in \mathbf{R} \sim \{0\}$. Then

$$S_c[C] = \{(y_1, \ldots, y_{n-1}, y_n) \in^R n : 0 \leq y_1 \leq 1, \ldots, 0 \leq y_{n-1} \leq 1,\ c y_{n-1} \leq y_n \leq 1 + c y_{n-1}\}$$

so

$$|S_c[C]| = \int_0^1 \Big(\int_{c y_{n-1}}^{1 + c y_{n-1}} \, dy_n \Big) \, dy_{n-1} = 1 = |\mathbf{det}\, S_c|.$$

Also,

$$D_c[C] = \{(y_1, \ldots, y_{n-1}, y_n) \in^R n : 0 \leq y_1 \leq 1, \ldots, 0 \leq y_{n-1} \leq 1,\ 0 \leq y_n \leq c\}$$

so

$$|D_c[C]| = |c| = |\mathbf{det}\, D_c|.$$

Finally, $P_\sigma[C] = C$ so

$$|P_\sigma[C]| = |C| = 1 = |\mathbf{det}\, P_\sigma|.$$