Clue 2: The degree of the maximal unramified abelian extension of $\mathbb{Q}(\sqrt{-39})$ divides n.

Answer 2: By Class Field Theory, the degree of the maximal unramified abelian extension of $\mathbb{Q}(\sqrt{-39})$ is the class number of $\mathbb{Q}(\sqrt{-39})$. There are several ways of computing this:

1. **The Minkowski Bound: [Cohn, Corr 1, p135]** To use the Minkowski bound to compute class numbers, we compute all possible (integral) ideals I of norm $N(I) \leq \sqrt{|-39|} < 7$. Every ideal class will contain such an ideal. By UPIF, it suffices to first determine the prime ideals $\mathfrak{p} \mid p \in \mathbb{N}$ of norm < 7, which we now do one prime at a time.
   First, we notice that the norm function on the ring of integers $Z[1, \omega]$ where $\omega = \frac{1+\sqrt{-39}}{2}$ is given by the quadratic form $x^2 + xy + 10y^2$. We can complete the square to write this as $(x+y/2)^2 + \frac{39}{4}y^2$, which means that is is impossible for the norm of any non-square to be less than 10.
   Step a: Determine the possible primes: $p = 2$**:** Since the Kronecker character $(-39/.)$ gives a character mod 39, at $p = 2$ it evaluates to $(-39/2) = (-39/(2+|-39|)) = ((2+|-39|)/-39) = (2/-39) = (2/39) = 1$ since $-39 \equiv 1 \pmod 8$, so $p = \mathfrak{p}_1\mathfrak{p}_2$ is split and there are 2 distinct ideals $\mathfrak{p}_i$ with $N(\mathfrak{p}_i) = 2$. Since the norm form does not represent 2, each of these primes live in non-principal ideals.
   $p = 3$**:** Here $(-39/3) = 0$, so $3 = \mathfrak{p}_3^2$ is ramified, and $N(\mathfrak{p}_3) = 3$. Since 3 is not a norm form, we see that $\mathfrak{p}_3$ is also a non-principal ideal.
   $p = 5$**:** Here $(-39/5) = (1/5) = 1$, so $5 = \mathfrak{p}_4\mathfrak{p}_5$ is split and each $N(\mathfrak{p}_j) = 5$. Here again the ideal are not principal.
   Step b: Determine the relations: Since $2 \cdot 5 = 10 = 0^2 + 0 \cdot 1 + 10 \cdot 1^2 = N(0 + 1 \cdot \omega)$ we see that $2 \cdot 5$ factors as $(\frac{1+\sqrt{-39}}{2})(\frac{1-\sqrt{-39}}{2})$. Thus we can take $\mathfrak{p}_1 = (2, \omega), \mathfrak{p}_2 = \bar{\mathfrak{p}_1} = (2, \bar{\omega}), \mathfrak{p}_4 = (5, \omega), \mathfrak{p}_5 = \bar{\mathfrak{p}_4} = (5, \bar{\omega})$. For convenience, we now compute $\omega^2 = \frac{-19+\sqrt{-39}}{2} = -10 + \omega$. With this we compute the powers of the ideal $\mathfrak{p}_1$ to see when it is principal. We already know that $\mathfrak{p}_1$ is not principal, so we compute

   $$\mathfrak{p}_1^2 = (2, \omega)(2, \omega) = (4, 2\omega, 2\omega, \omega^2) = (4, 2\omega, 10-\omega) = (4, 2\omega, 2-\omega) = (4, 2-\omega)$$

   where the last elimination follows since $2\omega = 2(2 - \omega) - 4$. If this was principal, then it would be generated by an element of norm 4, however the only such elements are ± 2 because those give the only

ways of representing 4 as $x^2 + xy + 10y^2$. Therefore $\mathfrak{p}_1^2$ is not principal since it cannot be written in the integral basis $[2, 2\omega]$.
We now compute

$$\mathfrak{p}_1^3 = (4, 2-\omega)(2,\omega) = (8, 4\omega, 4-2\omega, 2\omega-\omega^2) = (8, 4\omega, 4-2\omega, 10+\omega)$$

Since the norm of the ideal divides the GCD of the norms of its generators, when we have equality we can decide which elements to eliminate by eliminating those which are not essential for determining the norm of the ideal. In this case, the norms of the 4 generators are respectively: 2^6, $2^5 \cdot 5$, $2^4 \cdot 3$, and $2^3 \cdot 3 \cdot 5$. Therefore the norm 2^3 is the GCD of the norms of the first and last generators 8 and $10+\omega$. We then find that $4-2\omega = -2(10+\omega)+2\cdot 8$ and we can write 4ω as $(1+\omega)(10+\omega)-\omega\cdot 8$. Thus

$$\mathfrak{p}_1^3 = (8, 10+\omega) = (8, 2+\omega)$$

which is again not principal since it is not the ideal (8).
Finally we compute

$$\mathfrak{p}_1^4 = (8, 2+\omega)(2,\omega) = (16, 8\omega, 4+2\omega, 2\omega+\omega^2) =$$

$$(16, 8\omega, 4+2\omega, -10+3\omega) = (16, 4+2\omega, -10+3\omega)$$

since $8\omega = -4(10+3\omega)+10(4+2\omega)$. Thus we can simplify

$$\mathfrak{p}_1^4 = (16, 4+2\omega, 6+3\omega) = (16, 2(2+\omega), 3(2+\omega)) = (16, 2+\omega)$$

and the factorization $16 = (3-\omega)(2+\omega)$ shows that $\mathfrak{p}_1^4 = (2+\omega)$ is principal. Therefore the class group contains a cyclic subgroup of order 4, and the class number is divisible by 4.
To understand the ideal classes generated by the other possible prime ideals, we first notice that from the factorizations of 2 and 5 that the ideal classes of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are inverses, as well as the ideal classes of $\mathfrak{p}_4$ and $\mathfrak{p}_5$ are inverses of each other. Also from the factorizations of $10 = 2\cdot 5 = \omega\cdot\bar{\omega}$ into prime ideals, we see that $\omega = (2,\omega)(5,\omega) = \mathfrak{p}_1\mathfrak{p}_4$ gives the same relation for $\mathfrak{p}_1$ and $\mathfrak{p}_4$. Therefore we have only to analyze the ideal classes associated to the ramified prime $\mathfrak{p}_3$ over the prime $p = 3$.
The prime ideal $\mathfrak{p}_3$ so that $3 = (\mathfrak{p}_3)^2$ is given by $\mathfrak{p}_3 = (3, f(\omega))$ where $f(\omega)$ is the root of the minimal polynomial of ω (mod 3). Since $\omega^2 =$

$-10+\omega$ and $\omega \notin \mathbb{Q}$, we see that its minimal polynomial is $x^2 - x + 10$ which is congruent to $x^2 + 2x + 1 = (x+1)^2 \pmod 3$. Therefore $\mathfrak{p}_3 = (3, 1+\omega)$, and it gives an element of order 2 in the class group. Since we alrealy have the element $\mathfrak{p}_1^2$ as an element of order 2, we first check if it gives a new element of order 2 by checking that the product $\mathfrak{p}_1^2\mathfrak{p}_3$ is not principal. To do this we compute

$$\mathfrak{p}_1^2\mathfrak{p}_3 = (4, 2-\omega)(3, 1+\omega) = (12, 6-3\omega, 4+4\omega, 2+\omega-\omega^2) = (12, 6-3\omega, 4+4\omega, 12)$$

$$= (12, 6-3\omega, -8+4\omega) = (12, 3\cdot(2-\omega), -4\cdot(2-\omega)) = (12, 2-\omega) = (12)$$

because $(2-\omega)(1+\omega) = 12$. Thus $\mathfrak{p}_1^2\mathfrak{p}_3$ is principal, so $\mathfrak{p}_3$ gives the same ideal class as $\mathfrak{p}_1^2$. This means that we have already generated the entire ideal class group, which is a cyclic group of order 4, and the class number of $\mathbb{Q}(\sqrt{-39})$ is 4.

2. **Binary Quadratic Forms:** To compute the class number using binary quadratic forms, we use the fact that the class number of $\mathbb{Q}(\sqrt{-39})$ agrees with the number of reduced positive definite binary quadratic forms $Q(x,y) := ax^2 + bxy + cy^2$ over $\mathbb{Z}$ with discriminant $b^2 - 4ac = -39$. A form is said to be **reduced** if $|b| \leq a \leq c$ and if $b > 0$ whenever one of the inequalities is and equality. We also know that $0 < |a| < \sqrt{\frac{|-39|}{3}} = \sqrt{13} < 4$ for all such reduced forms. For each possible $1 \leq a \leq 3$ we compute the reduced forms, giving

$$a = 1 \longrightarrow b = 0, 1 \longrightarrow (a,b,c) = (1,1,10)$$

$$a = 2 \longrightarrow b = 0, \pm 1, 2 \longrightarrow (a,b,c) = (2, \pm 1, 5)$$

$$a = 3 \longrightarrow b = 0, \pm 1, \pm 2, 3 \longrightarrow (a,b,c) = (3,3,4)$$

Note that in each case we only need to consider b odd since -39 is odd. This gives 4 reduced forms, so the class number of $\mathbb{Q}(\sqrt{-39})$ is 4.

3. **The Class Number Formula (by evaluating the L-function at $s = 1$):**

4. **The Class Number Formula (by counting Quadratic residues and non-residues):** By the analytic class number formula, we know that the class number of $\mathbb{Q}(\sqrt{-39})$ is given as the number of quadratic residues minus the number of quadratic residues in the interval $1 \leq <$ ... References: Ireland and Rosen, ???; Washington's Book "Introduction to Cyclotomic Fields" Exercise ???.

Conclusion: The class number here is four, so this says $4 \mid n$.