

Clue 1: All of the prime divisors of (the mystery number) n are inert in the maximal totally real subfield of $K := \mathbb{Q}(\sqrt{-6}, \sqrt{-30})$, and at least one of its prime divisors ramifies in K .

Answer 1: Since the biquadratic extension K/\mathbb{Q} is a Galois extension of degree 4 with Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the Main Theorem of Galois Theory tells us that it contains exactly three intermediate subfields all of which have degree 2 over \mathbb{Q} . It is easy to see that $\sqrt{-6} \cdot \sqrt{-30} = \sqrt{5} \in K$, hence $\mathbb{Q}(\sqrt{5})$ is the totally real subfield of K .

This means that all of the prime divisors p of n are inert in $F = \mathbb{Q}(\sqrt{5})$, hence

$$p \mid n \quad \Rightarrow \quad \left(\frac{5}{p}\right) = -1 \quad \Rightarrow \quad p \equiv 2, 3 \pmod{5}.$$

In particular, $5 \nmid n$.

We also know that some $p \mid n$ ramifies in K , which means that some $p \mid \text{Disc}(\mathbb{Q}(\sqrt{-6})) = -2^3 \cdot 3$ or $p \mid \text{Disc}(\mathbb{Q}(\sqrt{-30})) = -2^3 \cdot 3 \cdot 5$. Since $5 \nmid n$, we see that n is divisible by either $p = 2$ or $p = 3$ (both of which are inert in F).